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The Impact of Erroneous Examples on Students' Learning of Equation Solving

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Equation solving is regarded as a gatekeeper for pupils who wish to study higher level algebra courses. When first learning algebra, students tend to acquire some misconceptions. It is important to help them discover and remediate such misconceptions. This study focuses on using erroneous, incorrectly worked examples to confront students' own possible underlying misconceptions. The results indicate that when teaching algebraic equations, combining the use of incorrect examples and correct examples leads to the improvement of algebraic conceptual understanding as compared to using two correctly worked examples. Different presentation patterns of the combination of correct and incorrect examples show no significant difference. The students with a higher level of prior knowledge of mathematics performed better than students with lower level prior knowledge.

Key words: incorrect examples; equation solving; algebra.

Researchers have extensively documented that algebra is a "gatekeeper" to more advanced mathematical topics (Lange & Booth, 2015). Mathematics teachers know this from their teaching experiences. The solving of equations is usually learned at the very beginning stages of learning algebra. Students should have a strong understanding of equation construct in order to be successful in learning more complex algebraic concepts. Unfortunately, some research studies found that if students receive very poor preparation in solving equations, they will not have accurate conceptual knowledge about the critical features in equations. For example, they often use the distributive property incorrectly; for instance, when presented with $2+3(x+1) = 8$, students make mistakes when they distribute 3 by simply removing the parentheses: $2+3x+1 = 8$.

Misconceptions lead students to misunderstand procedures for solving equations correctly, and these difficulties in solving equations hamper students from developing a deeper understanding of algebra (Booth, Kenneth & Newton, 2013). A study by An, Kulm, and Wu (2004) echoes the need for pedagogical content knowledge (PCK) stating that the primary knowledge needed by mathematics teachers is pedagogical content knowledge, and one dimension of pedagogical content knowledge is addressing students' misconceptions. With strong PCK, a mathematics teacher will help students build a strong and accurate conceptual understanding of algebra.

Although in recent years there is a growing body of literature focused on mathematical misconceptions (Borasi, 1987; 1994), important questions, such as "How can mathematics teachers utilize more effective methods to help students build strong and more accurate equation knowledge?" and "How can mathematics teachers help correct commonly held misconceptions in students' minds in order to increase students' success in algebra?" have not been addressed.

Some studies in mathematics education have explored the phenomenon of erroneous examples (incorrectly worked examples) and provided anecdotal evidence that studying errors can help students' learning. Borasi (1987) argues that mathematics education could benefit from the discussion of errors by encouraging critical thinking about mathematical concepts, by providing new problem solving opportunities, and by motivating reflection and inquiry. Some researchers have demonstrated that learning equations is improved when students learn the combination of a correctly worked example and an incorrectly worked example in the context of technology-enhanced learning (TEL), or in a classroom setting through a computer program (Booth et al., 2013). Utilizing a computer based system, some researchers have tested the theory that studying erroneous examples in the domain of fractions can help students learn from the common errors of other students (Tsovaltzi, Melis, Bruce, Meyer, Dietrich & Goguadze, 2010). An incorrectly worked or erroneous example is a step-by-step description of how to solve a problem in which one or more of the steps are incorrect. They have argued that confronting students with mathematical errors can be valuable, particularly when students are sufficiently prepared to deal with errors.

On the other hand, many mathematics teachers in traditional classrooms are reluctant to discuss errors with students because they worry that allowing errors to appear, or exposing learners to errors, may confuse students' learning, presumably because these errors are then more likely to be retrieved during later problem solving. The teachers recall the studies of Skinner and other behaviorists which showed that the lack of a "penalty" when undesirable behavior was exhibited led to the repetition of the behavior (Skinner, 1968).

This study will focus on the following two research questions: (1) Will the use of an incorrect example benefit students who are learning to solve equations? (2) If the incorrect example is helpful, will it be beneficial for all students?

First, we hypothesize that the use of incorrectly worked examples will be beneficial for learning not only in an e-learning environment, but also in a traditional classroom. Any supposed disadvantage when using errors as examples can be avoided if the incorrect examples are presented appropriately. Using incorrectly worked examples can be a new tool for teachers when teaching equation solving and students will be able to understand the equation solving process at a deeper level.

Second, we hypothesize that only the students with a high level of prior knowledge will be able to learn adequately from the use of erroneous examples.

Literature Review

Equation Solving

Because of its step-by-step procedural nature, solving equations as an analytic, non-contextual task monopolized algebraic manipulations and was often subject to rote learning (Michal & Shoshana, 1997). Nonetheless, solving equations is a pivotal procedure to learn in beginning algebra, one that is often confusing and challenging for many students. Many misconceptions that prohibit students from developing a deeper understanding of algebra occur at this point in their learning and persist from then on (Lange, Booth & Newton, 2014). For example, students often incorrectly solve the equation "$3x - 4 = 5$" by subtracting 4 from both sides, which results in the incorrect answer "$3x = 1$." By explaining how the application of an incorrect procedure leads to an incorrect answer, students are forced to accept that the procedure is wrong and to notice that the negative sign that precedes the 4 makes it inappropriate to apply this strategy. Such misunderstandings, if they persist, have been found to have a detrimental effect on students' ability to solve equations and also hinder students' ability to learn new algebraic content (Booth & Kenneth, 2013).

Worked Example

A worked example consists of a problem, the solution steps, and the final solution itself. It is primarily used in well-structured domains such as mathematics and physics. A worked example in mathematics is a problem that has been fully completed to demonstrate all the procedures necessary to acquire its solution (Clark, Nguyen & Sweller, 2011). Zhu and Simon (1987) found that their sequenced mathematical examples were sufficient to induce skills and abstract problem representations without providing explicit instruction. This has become known as ''the worked example effect'' (Nelleke, Tamara & Saskia, 2012).

Further studies discovered that because worked examples typically do not include all the reasons why a certain step in the solution was performed, unless they are well-designed, the use of worked examples does not guarantee good learning results. That is to say, the design of the examples plays a crucial role in their effectiveness (Tarmizi & Sweller, 1988). Later research was dedicated to discovering how to design appropriate examples. Catrambone (1996;1998) has proposed that formatting an example's solution to accentuate its sub-goals, either by affixing a label to them or by simply visually isolating them, can assist a learner in actively inducing the example's underlying goal structure and this can guide a learner to discovering useful generalizations. Reed and Bolstad's study (1991) indicates that two examples can facilitate learning better than can a single example.

Erroneous Examples

Research in mathematics education has explored the phenomenon of erroneous examples and provided anecdotal evidence that studying errors can help student learning (Borasi, 1987). One of the effective ways of promoting comprehension in learning from worked examples is using incorrect examples. Some researchers believe that learning from errors can help students develop (or enhance) their critical thinking, error detection, and error awareness skills (Baker, Corbett, Koedinger, & Wagner, 2004). Tsovaltzi et al. (2010) found that erroneous examples could help students improve both their cognitive and metacognitive skills.

The erroneous examples were developed using an intelligent tutoring system (ITS) which supports rapid development of web-based learning materials and produces exercises that track and log students' interactions with the software. Relying on the intelligent technology of a self-paced intelligent tutor system, Booth et al. (2013) showed that students perform better overall on equation-solving posttests after receiving both correctly worked and incorrectly worked examples (rather than receiving only correctly worked examples).

Can students in a traditional classroom display better conceptual understanding after completing assignments that include analyzing both correct examples and incorrectly worked examples? This study explored an effective approach to teaching pupils equation solving by using incorrectly worked examples. In the proposed study, incorrectly solved equation examples were combined with correctly worked examples. We hypothesized that learning by using the incorrectly worked examples forces students to pay attention to the actions which resulted in the wrong answers. When comparing the correct and incorrect examples, students will think more about the correct concepts or the correct steps that needed to be carried out. Therefore, the use of incorrectly worked examples will benefit pupils' learning if teachers present them carefully in their instructional activities.

Prior Knowledge

In addition to design, students' prior knowledge also plays a role in the effectiveness of instruction using worked examples. Some empirical research found that only advanced students could benefit from the inclusion of incorrectly worked examples; students with low prior knowledge learned more from the use of correctly worked examples (Grosse & Renkl, 2007). Other research, however, has produced contrary results in this area (Tim & Heinze, 2013).

The authors hypothesize that erroneous examples will be more beneficial to students with high prior knowledge because understanding the incorrectly worked equations requires a solid mathematical foundation. Students with minimal prior knowledge in mathematics will not grasp the appropriate strategies or concepts from studying incorrectly worked examples.

Methods

Subjects

The subjects were 90 fourth grade students from a primary school, 46 boys and 44 girls in Dalian, China. They were approximately ten years old. Their teachers were asked to evaluate the students utilizing their in-class test scores and teacher observation as to their level of prior knowledge, categorizing them as having high prior knowledge or low prior knowledge. The children in each of these prior knowledge subgroups were then randomly assigned to one of the three experimental study groups, Condition 1, Condition 2, and Control group, so that each study group contained 15 students with high prior knowledge and 15 students with low prior knowledge. All the students were novices in respect to solving equations.

Of the three experimental study groups, two groups used both correctly worked examples and erroneous examples, and one group used only correctly worked examples. The two groups using incorrect examples were characterized by two different conditions: the Condition 1 group used one correct example plus one incorrect example; the error in the second example was notated with an "x" and the corrected step was displayed. This group was called IEIC. The Condition 2 group used one correct example plus one incorrect example; the error was also notated with an "x" but this time an explanation was provided as to how to correctly solve the equation. This group was called IEIE. The third group was the control group which used two correct examples with an explanation stating how to correctly solve the equation. Each group had 30 subjects.

Materials

In the experiment, the authors dealt mainly with the distributive property in equation solving. The correct example was a problem displaying the correct solution steps. The incorrectly worked example included only one "false" step which used the distributive property incorrectly. For example, if the equation was: 3+2(x+1) = 7, the incorrect step was: 3+2x+1 = 7.

There were three sets of learning materials, each containing a different combination of worked examples. The first set of materials consisted of an equation which was correctly solved and displayed all the steps showing how the solution was obtained, but provided no explanation. The second equation was incorrectly worked and an "×" indicated the wrong step in the solution process. The correct step could be found at the bottom right of the paper and it was identified as the correction to the error in the equation. Condition 1 IEIC students read the correct solution for the example carefully and corrected the equation using the correct step provided at the bottom right of the paper.

The second set of materials consisted of an equation which was correctly solved and displayed all the steps showing how the solution was obtained, but provided no explanation. The second equation was worked incorrectly and an "×" indicating the wrong step in the solution process. An explanation was then provided to explain how to expand an equation when using the distributive property. Condition 2 IEIE students read the correctly worked equation; then they read the incorrectly worked equation and the explanation as to how to properly expand an equation, which should explain why the identified step was incorrectly worked.

The third set of materials consisted of two correct examples with an explanation as to how to correctly expand an equation. The control group students read the two examples and the explanation on the bottom carefully.

All the examples were of equation solving problems each consisting of four steps. All materials were printed on A4 paper. Students could make notes on the paper. (See Appendix)

Procedures

Pretest stage (5 minutes). All students completed the pretest at the same time, but worked individually.

The study stage (10 minutes). Following the completion of the pretest, students studied the learning materials for their specific group for 10 minutes. Then all students handed in their learning materials.

The test stage (15 minutes). After 5 minutes of time to rest, students took a posttest.

Instruments

Pretest materials consisted of one equation solving problem similar to the example learning materials, printed on A4 paper. Only the students who did not solve this equation correctly would be allowed to participate the next two procedures. Otherwise they were excluded from the study. Learning materials are described above. Posttest materials consisted of five equation solving problems similar to those in the worked examples, but with different numbers, printed on A4 paper.

Each correct solution received 3 points; partially correct solutions received 2 points or 1 point; totally wrong solutions received 0 points.

The authors regard the test score as an independent variable. Learning conditions and prior knowledge are regarded as dependent variables in analyzing and interpreting the data. The data were analyzed using 3 (Learning Condition) ×2 (Levels of Prior Knowledge) UNIANOVA test. The pretest and posttest questions were designed and pilot tested carefully. Prior to pilot testing, they were discussed with colleagues in the same research arena and with elementary school teachers who teach equation solving.

Results

In examining the learning effect of the incorrect example, the authors compared the three conditions of the experiment. To examine the role of prior knowledge, the authors also compared the two different levels of prior knowledge. In addition, the authors also examined the interaction effect between the three conditions and the two levels of prior knowledge in order to explore if only the high level prior knowledge students demonstrated a high achievement after receiving incorrectly worked example learning materials.

The results from a three-way ANOVA are shown in Tables 1 and 2. The significant differences between the three conditions can be found in Table 1, and the means and standard deviation of test score results can be found in Table 2.

Table 1
UNIANOVA of Erroneous Example Learning Effect

Independent variable	*Source of variation*	*SS*	*df*	*MS*	*F*	*ηp2*
Test Score	Learning Condition	45.356	2	22.678	5.115**	0.109
	Prior Knowledge	1424.044	1	1424.044	321.213***	0.793
	Learning Condition × Prior Knowledge	0.822	2	0.411	0.093	

Note：$^{**}p<0.01$；$^{***}p<0.001$

Table 1 indicated that the mean effect of the learning condition was significant, $F(2,88)=5.115$，$p<0.01$. The mean effect of prior knowledge was significant, $F(1,88)=321.213$，$p<0.001$. The results supported our hypothesis #1, that students with high prior knowledge benefited more from the incorrect examples. The interaction of learning and prior knowledge was not significant.

Based on the Univariate Analysis of Variance, we used Least Significant Difference (LSD) to analyze the difference between the different learning conditions and different prior knowledge levels separately (see Table 3). The posttest scores from high to low in turn were IEIC, IEIE and control group (8.50>7.97>6.80).

Table 2

Learning Effect of Each Condition

Group	IEIC		IEIE		Control Group	
	M	*SD*	*M*	*SD*	*M*	*SD*
High	12.60	1.805	11.933	2.404	10.67	2.664
Low	4.40	1.882	4.00	2.138	2.933	1.534
Sum	8.50	4.547	7.97	4.612	6.80	4.475

Table 3

LSD of Different Learning Conditions

Learning Group（*I*）	Contrast Group（*J*）	（*MD*）
Control Group	IEIC	-1.70*
	IEIE	-1.16*
IEIC	IEIE	0.53

Note：*MD=I-J* $^{*}p<0.05$

Table 2 and Table 3 show that the difference between both erroneous example condition groups and the control condition group was significant. The results manifested that a combination of erroneous example and correct example presentations resulted in a better learning effect than presentations with two correct examples. These results supported our first hypothesis.

In addition, the learning effect between two incorrect example conditions was not significant, indicating that there were no differences detected among the two methods of presentation.

Discussion

In summary, our findings indicate that it can be effective for learners with favorable prior knowledge to consider typical errors when learning to solve algebraic equation problems. Therefore, the ‘‘fear’’ that learners who are confronted with incorrect solutions might acquire incorrect procedural knowledge instead of learning why the error occurred is unfounded. The employment of incorrect solutions in real classrooms for learners with good foundational understanding and prior knowledge might enrich and enhance their mathematical education.

The results from this study indicated that studying with incorrect examples was effective. There are several possible explanations for the results. An error in an erroneous example can serve as a springboard for inquiry (Borasi, 1994). Highlighting errors in worked examples helps students learn from those errors. As Star and Seifert (2006) concluded, learning to use the distributive property is the first step in equation solving. Many students do not have sufficient algebraic thinking knowledge about the critical features in equations, and so they always expand the equations with parentheses like this: 2(x+1) = 2x+1, probably because they are used to whole number operations, such as, a number multiplies a number. When a number is multiplied in an algebraic expression, they forget that they must multiply both the items in the parentheses separately with the number preceding the parenthesis. Incorrect examples remind pupils to notice the forgotten item. One student who correctly answered the problems in the later exam stated the following:

At the beginning, I read one example. I knew that solving an equation needs some steps. Then I read another example, I noticed that there was a rubric "✘" following a step. So I think, why was it wrong? I went back to the first example. I got it! When I do the multiplication in the first step, I should multiply the two items in the parentheses, because they are together.

From the experiment conducted in this study, we concluded the following: first, prompting students to learn and reflect on errors will lead them to think more carefully and deeply when using the distributive property.

Second, comparing incorrect examples to correct examples can draw students' attention to the particular features that make the procedure inappropriate in a problem, and can then help the students replace faulty conceptual knowledge with correct conceptual knowledge about equations. Combining a correct and an erroneous example together will effectively support student learning.

In the experiment in this study, both the correction and the explanation of the error forced students to think about the steps that had been carried out and the reasons why these actions were wrong and then to confront their own possible underlying misconceptions. The two different ways of presentation both had a positive effect on learning.

Conclusion

This study extended prior research that found that incorrect examples can be used effectively in a real classroom. First, the results of this study show that using the combination of a correct example and an incorrect example leads to improved conceptual understanding as compared to using two correct worked examples, and that different presentation patterns of the combination of correct and erroneous examples shows no significant difference in student learning. Thus, using incorrect examples is a strategy to support student understanding and success in equation solving. By using the tool of incorrect examples, teachers will be successful in targeting their students' misconceptions or error solving processes in solving equations. When teaching equation solving, incorrect examples can be included as a part of the teaching methodology. The best way of presenting incorrect examples in class is to combine incorrect and correct examples together, so that students can notice the error concept by comparing and making sense of the correct concept.

Second, students with high levels of prior knowledge performed better than students with low levels of prior knowledge when presented with learning that included an incorrect example. Using incorrect worked examples in class, teachers should give more instruction to students who demonstrate a lower level of understanding because they display more difficulties in localizing, understanding and correcting the errors.

Limitations and Recommendations

This study explored the use of erroneous examples as a teaching tool in students' learning of algebraic equation solving. The present experiment was conducted with the combination of a correctly worked example and an incorrectly worked example. Other forms of incorrect examples were not included. It is not clear whether the results apply to other content areas of mathematics.

Future research should address the following areas: The first area needing research is: Can erroneous examples be used effectively with other mathematical strategies or concepts in a traditional classroom? The second area needing research is: What other forms of incorrect examples can be designed to enhance students' learning mathematics? A third area needing research is: How much prior knowledge do students need to have before the use of erroneously worked examples becomes beneficial for them?

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Appendix

Condition 1: IEIC	Condition 2: IEIE	Control group
Correct Example: 3+2(x+1)=7 3+2x+2=7 5+2x=7 2x=2 X=1	Correct Example: 3+2(x+1)=7 3+2x+2=7 5+2x=7 2x=2 X=1	Example 1: 3+2(x+1)=7 3+2x+2=7 5+2x=7 2x=2 X=1
Incorrect Example: 1+2(x+1)=9 1+2x+1=9 × 2+2x=7 2x=5 X=5/2 The correct step is: 1+2x+2=9	Incorrect Example: 1+2(x+1)=9 1+2x+1=9× 2+2x=9 2x=7 X=7/2 When we expand the parentheses, use the number to multiply all the items in it.	Example 2: 1+2(x+1)=9 1+2x+2=9 3+2x=9 2x=6 X=3 When we expand the parentheses, use the number to multiply all the items in it.

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