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Survey of Primary Students' Mathematical Representation Status and Study on the Teaching Model of Mathematical Representation

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Mathematical representation is an instrument for students to grasp mathematical knowledge and ability. Nowadays, in problem solving, primary students of different grades show different trends of the representation model, and mastering mathematical language is a root cause affecting students' mathematical representation. The problems in the teaching of primary mathematical representation are as follows: teachers fail to pay enough importance to mathematical representation; the teaching quality of mathematical representation is not satisfactory; students have barriers in the understanding and conversion of mathematical language; mathematical communication ability needs improvement. As a result, understanding the value of teaching mathematical representation, combining the relations between mathematical representation, mathematical language and mathematical communication and constructing the teaching model of primary mathematical representation are very important.

Key Words: mathematical representation, mathematical language, mathematical communication, teaching model

In the information society with the rapid development of science and technology, people need the ability to use and deal with language in order to express their ideas. Mathematics is an academic discipline used to convey knowledge and meaning through language based upon mathematical activities, such as the study of concepts, formulas, principles, methods and problem solving. The structure of language in mathematical activities includes external communication such as written and oral representation of symbol, word, graphics, and images (Li, 2001). Mathematical representation is carried out by means of these external communications. In fact, representation points to both process and result (Denisse, Thompson, Michaele, & Chappell, 2007). This study mostly focuses on the former and defines Mathematical representation as language activities——describing mathematical phenomenon, communicating mathematical ideas, solving mathematical

problems by written and oral representation of words, symbols, and graphics as a carrier.

Mathematical representation plays a vital role in mathematical activities. Mathematical representation ability as an important component of mathematical literacy has become the educational aim in many countries. "Principles and Standards for School Mathematics" proposes ten criteria, one of which is "mathematical representation". Instructional programs from prekindergarten through grade 12 should enable all students to create and use representations to organize, record, and communicate mathematical ideas; select, apply, and translate mathematical representations to solve problems; use representations to model and interpret physical, social, and mathematical phenomena (NCTM, 2000). Furthermore, "Mathematical Curriculum Standards for Full-time Compulsory Education" calls for all students to make a reasonable interpretation of relevant digital information in real life, solve simple problems with numbers, letters and graphs, describe the real world with mathematical symbols and graphics … establish the initial number sense and symbol sense which is used mainly to represent and communicate with number, understand number relationships represented by symbols, and convert between the symbols (Ministry of Education, PRC, 2001). Moreover, many studies at home and abroad have shown that mathematical representation is an instrument for students to grasp mathematical knowledge and skills, as well as an important criterion for teachers to assess students' mathematical ability and level.

Successfully arriving at solutions to mathematical problems utilizes a combination of problem representation skills and symbol manipulation skills (Brenner et al., 1997). The former involves skills that "include construction and using mathematical representations in words, graphs, tables, and equations". Another study shows that the methods of mathematical representation in problem solving are usually divided into five areas (Shirley, 2006): numerical representation, graphical representation, verbal representation, symbolic representation, dual representation. Numerical representation focuses on specific numerical values in a variety of formats, such as decimal, fraction, or percent; or a numerical list, such as a list of numbers appearing as outcomes of probability. Graphical representation contains six distinct visual representations - pictorial, model, horizontal charts, vertical charts, graphs, and coordinates graphs. Pictorial representations use real-world objects such as toys. Models refer to an object used to represent something else, such as cups and counters representing an algebraic expression. Verbal representation requires the use of written language to understand, describe, analyze, explain, or reflect upon numerical, algebraic, or graphic representation that does not include brief phrases such as directions for solving the problem. Symbolic representation focuses on symbolic notation and includes the use of variables and formulas. Five symbolic representations are found—equation, expression, algebraic equation,

algebraic expression, and formula. Dual representation contains two of the above listed representations categories and seven different combinations of representations are noted. Dual representations appear inside dashed boxes, and arrows indicate representational categories that are linked by the dual representation (See figure 1). This study is based on the results of previous research and focuses on the following issues: surveying primary students' mathematical representation status, such as the types and characteristics of mathematical representations used in problem solving, and trying to build a teaching model of mathematical representation.

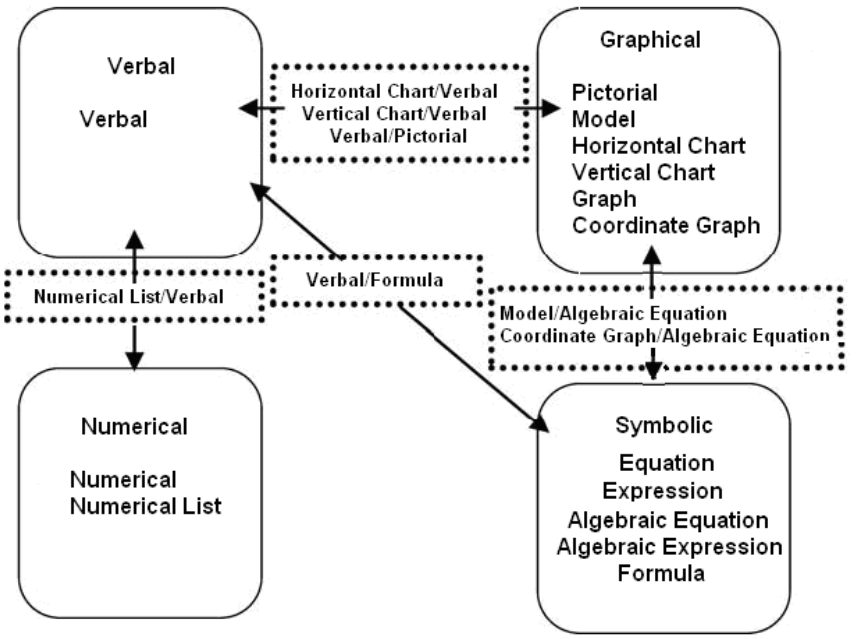

Figure 1: *Dual representation.*

Designs and Implementation

Object

In this study, sample surveys and interviews were carried out among students and teachers from grade 3 to grade 6 in a Guangzhou primary school. One hundred eighty questionnaires were sent to students and 174 valid questionnaires are returned. Twenty nine teachers, whose average teaching was for 14 years, were investigated.

Tool

The students' questionnaire, which included five algebra problems,

was used to survey the types and problems of mathematical representation in problem solving. There are two reasons why this study examined algebra questions: (1) all students should learn algebra, and algebraic literacy is very important to students' future work or further education, (2) algebra problems require students to understand and use various representations, so this study would get abundant statistical materials. The contents of the teachers' survey and interview addressed their standpoints about the importance of mathematical representation, factors affecting mathematical representation, their experience and difficulties in the practical teaching of mathematical representation.

Data

Student surveys were administered in class, and teachers after class. After excluding invalid questionnaires, data was analyzed using Excel. According to the statistical results, some teachers and students were interviewed.

Results and Analysis

Primary Students' Mathematical Representation Status in Problem Solving

Primary students of different grades show a difference on mathematical representation in problem solving. Although all students solve the same problems, Grade 3 and grade 4 students tend to use numerical representation and graphical representation. With an increase in grade, the probability of using verbal representation, symbolic representation and dual representation gradually increased (See table 1). Numerical representation and graphical representation are intuitionistic and concrete. Symbolic representation with specific abstraction and generality is mathematics exterior representation and relatively difficult to grasp and use (Li, 2001). Students' concrete thinking will also gradually run to abstract thinking, the knowledge needed to learn will continually expand and deepen. So, the changes showed in table 1 are in accord with the characteristics of the development of student thinking and practical teaching.

Table 1

The Types of Students' Mathematical Representation in Problem Solving

Type	Grade 3	Grade 4	Grade 5	Grade 6
	n=40×5 %	n=45×5 %	n=52×5 %	n=37×5 %

Numerical	89	44.5	78	34.7	50	19.2	36	19.5
Graphical	35	17.5	28	12.4	15	5.8	13	7.0
Verbal	10	5.0	21	9.3	28	10.8	18	9.7
Symbolic	8	4.0	15	6.7	77	29.6	42	22.7
Dual	0	0	10	4.4	24	9.2	45	24.3

Note:
1. Forty grade 3 students were investigated. Everyone solved five problems. "n=40×5" means that 40 students solved "40×5" problems. Among all problems, 89 were solved with numerical representations, and the proportion of the total is 44.5%. The rest may be deduced by analogy.
2. The data in table1 only involve the number of correct answers.
3. If the problems are represented by two types of representations, they will only belong to dual representation. If the problems represented by three or more types of representations, they will be respectively counted.

In addition, we can find that, the senior students still tend to use symbolic representation, although the problems can be solved with numerical or graphical representation. Most of senior student interviewees do not deny the correctness of numerical and graphical representation, but some say they should learn to use new knowledge, otherwise new knowledge make no sense. Some think symbolic representation will be more accurate and convincing in problem solving, and some believe that filling-in-blanks or multiple-choices can be solved with numerical or graphical representation, but in order to get full credit, other questions should be neatly solved with symbolic representation. However, individual students are not sure of the correctness of numerical or graphical representation. If some questions can be solved by both equation and arithmetic formula, these students are unwilling to use the latter, because they have such experience - when they use arithmetic formula to solve similar problems, their teachers tell them not always to use the previous approaches once they have learned equations. In conclusion, the characteristics of the primary students' mathematical representation and their ideas are very useful in practical teaching.

Mastering mathematical language is a root cause affecting students' mathematical representation. After carefully analyzing all aspects of written examination, we can find that students' failure in the examination is not due to failure to grasp the relevant algorithms, but with language difficulties (Zheng, 2004). Our results also indicate that students' comprehension, conversion and representation abilities of mathematical language are the root cause affecting their appropriate mathematical representation. When solving mathematical problems, students cannot extract the necessary mathematical information and do not understand the mathematical language in the problems, appear incompetent on language conversion, and so they are unable to represent and solve the problems successfully. In addition, teachers believe that in problem solving, the factors

affecting mathematical representation include students' knowledge structure, mastery of mathematical language, usage of mathematical metacognitive-strategy, logical reasoning ability, written representation ability of mathematical language, time and environment of mathematical problem solving, subject material contents, and so on. 86.2 and 82.8 percent of teachers respectively think that students' mastery of mathematical language and logical reasoning ability are very important factors, while 93.1 percent of teachers agree that the former is the most fundamental reason. From the theories of information processing, the process of problem solving is apperceiving and obtaining imagery firstly, mobilizing the existing knowledge and information to process the imagery in a certain logical order secondly, and logically representing the process of thinking activities with written language in the end. In short, mathematical problem solving is a course of comprehension, selection, processing, conversion and representation of mathematical language. If students have barriers in comprehension and conversion of mathematical language, correct representation cannot be carried out smoothly.

The Problems in the Teaching of Primary Mathematical Representation

Teachers fail to attach enough importance to mathematical representation. In the investigations, 96.6 percent of teachers agree that mathematical representation is a kind of language activity. 51.7 and 44.8 percent of teachers respectively think that mathematical representation plays a very or comparatively important role in students' learning. But only 27.6 percent of teachers really pay attention to students' mastery of mathematical language, and only 10.3 percent of teachers consciously carry out the teaching of mathematical representation to foster students' representation ability. Few teachers study mathematical language and representation pedagogy. As a whole, 72.4 percent of teachers consider that few teachers think much of mathematical representation under the current educational system. This shows that teachers don't attach enough importance to students' mathematical representation ability.

The teaching quality of mathematical representation affected by many factors is not satisfactory. Our results show that the factors affecting mathematical representation teaching include the following. With regard to teachers, on the one hand, their ability of controlling mathematical language is limited. Some teachers express they can't deeply understand some mathematical language, sometimes their representations are ambiguous, so their guidance for students is greatly weakened. On the other hand, teachers don't clearly understand the value of mathematical representation teaching. Most teachers believe that they communicate with students by mathematical language everyday, so it is not necessary and significant to carry out the teaching of mathematical representation. In addition, these teachers have no strong desire to continuously study mathematical education theories. They

don't understand the connotation, characteristic, procedure of mathematical representation teaching. As for students, learning differences, mastery of mathematical language, mathematical communication ability, interest in learning mathematics, psychological fear of teachers and classmates, written and oral representation ability of mathematical language, and so on, are all the factors affecting representation. Other factors include the pressures of teaching task and examination competition, school management system, chances of representation, learning atmosphere, and experts' guidance. Overall, the status of mathematical representation teaching affected by many factors is not satisfactory.

Mathematical language teaching should be strengthened. From the statistical results, we can find that many students whose mathematical representation ability is poor have barriers to understanding and translating mathematical language. For example, they cannot completely comprehend mathematical words, cannot grasp the relationships between mathematical languages, and cannot translate graphical language into verbal language, or verbal language into symbolic (numerical) language. These barriers directly influence students' mathematical representation. They cannot represent at all, or they represent inaccurately, or they cannot represent distinctly and roundly, or they are unable to represent the same mathematical information with different language types (Wu, 2002). In teaching, we must strengthen the understanding and conversion of language, not only pay attention to the longitudinal relationships of mathematical language, but also horizontal conversion (Zhuan Sun, 2002). So, fostering and improving students' representational ability cannot be separated from mathematical language teaching.

Mathematical communication ability needs improving. Mathematical oral and written representational ability can be defined as the ability of communicating mathematical thinking through mathematical language. The cultivation of mathematical representational ability cannot be separated from the mastery of mathematical language as well as mathematical communication ability. But our results show that 86.2% of teachers say very few students can exchange ideas by reasonable representation ways when talking about mathematical problems. Besides a few student interviewees which seriously ponder problems and communicate with others, the else don't take part in talking. The reason is that some students are afraid of being laughed at, some don't know how to express their ideas, and some are unwilling to independently think just waiting for the teachers' explanation. Thus, students' performance of communication and cooperation with others is not satisfactory. They should get more encouragement and guidance.

Conclusions and Recommendations

Through the investigations, we can find that in problem solving, primary students of different grades show different trends of the representational model, and mastering mathematical language is a root cause affecting primary students' mathematical representation. At the same time, the problems in the teaching of primary mathematical representation are as follows: teachers fail to attach enough importance to mathematical representation, the teaching quality of mathematical expression is not satisfactory, students have barriers in the understanding and conversion of mathematical language, and students' mathematical communication ability needs improving. These findings will bring some references to practical teaching. We will further elaborate the value of mathematical representational teaching, analyze the relations between mathematical representation and other concepts, and try to construct a teaching model of primary mathematical representation and present some teaching suggestions.

The Educational Value of Mathematical Representation Teaching

In daily teaching, teachers should recognize that cultivating students' mathematical representational ability is not only the goal of the curriculum standard, but an important aspect of improving students' mathematical literacy. The value of mathematical representational teaching is as follows:

To help students consolidate their knowledge and improve skills. Learning mathematical knowledge is one goal of mathematical teaching, such as definitions, theorems, formulae, mathematical knowledge. So learning mathematical knowledge essentially is equivalent to grasping mathematical language. In mathematical representational teaching, students are constantly led to express their own ideas in mathematical language and encouraged to explain, describe the mathematical problems by reasonable representational models. Their understanding of mathematical language is continually deepened； mathematical knowledge is consolidated, and cognitive level and mathematical ability are developed.

To help teachers and students enrich their concept of mathematics and mathematical teaching. Sometimes language in mathematical teaching is the reflection of the concept of mathematics and mathematical teaching. Mathematical learning should be based upon independently thinking (Zheng, 2004). In mathematical representational teaching, "mathematically talking" and "mathematically writing" as language activities are constantly carried out. Both written representation and oral representation all rely on mathematical language. Teachers and students can feel that mathematics should be regarded as a kind of language by which our world can be scientifically described, and mathematical learning not only relies on independently thinking, but "thinking

aloud" and " learning from speaking out" are all important teaching methods.

To help students overcome the psychological barriers. For a long time, students firmly believed that mathematics was difficult to learn. In addition, some were afraid of being laughed at or criticized, and some had inferiority complexes, etc. The psychological barriers hindered students in their learning of mathematical language and representation. In mathematical representational teaching, teachers should strive to create a democratic, equal, harmonious teaching atmosphere and provide learning opportunities for cooperation and communication, provide students with a platform to represent, provide proper guidance, and enhance students' self-confidence and courage. These strategies can help students overcome the psychological barriers.

To help teachers assess students' learning result. Mathematics curriculum reform brings forward new assessments. Traditional paper-pencil exercises are no longer the only assessment means. In mathematical representational teaching, students have more opportunities to express their opinions through mathematical language. So, teachers can listen to students respectfully, may have a new understanding about students, and obtain new valuable evidence to assess students' learning.

To help teachers improve their own literacy. Nowadays, teachers are acting as the organizer, guide and copartner in mathematical activities, and should have better mathematical literacy. In mathematical representational teaching, teachers need to deeply understand and accurately represent verbal, symbolic, graphical language, and possess the ability to control mathematical language. They should be good at organizing teaching through mathematical language, leading students to express themselves freely in cooperation and communication, and giving students reasonable assessments. During the course of teaching, teachers' literacy will be greatly improved.

The Relationships between Mathematical Representation and Mathematical Language, Mathematical Communication

Mathematical language is a kind of scientific language which can be divided into verbal language, symbolic language and graphical language. Mathematical representation is the written and oral language activity which is based on the mastery of mathematical language, whereas mathematical language can be learned, used and tested in representation. So, mathematical representational teaching is useful to consolidate and develop mathematical language. In a word, mathematical language and mathematical representation are interdependent and mutually reinforcing.

International mathematical education focuses on mathematical communication and representation which are the important components of mathematical ability and means of learning. Mathematical communication is a kind of dialogue between students and teachers, students or mathematics itself one part of which can be regarded as mathematical language activities. The

degree of communication can be greatly expanded by means of mathematical representation. On the other hand, the cultivation and development of mathematical representational ability needs a good communication environment. So, in short, mathematical representation and mathematical communication can be carried out simultaneously and can encourage each other.

In summary, there is a close connection between mathematical language, mathematical communication and mathematical representation. The first two concepts are indispensably important elements in mathematical representational teaching which will be presented in the teaching model of mathematical representation.

The New Teaching Model of Mathematical Representation

At present, there is little research about mathematical representational teaching. One study actively explores and summarizes an effective teaching method——primary teaching method of mathematical representation which has great important teaching value (Zhang, 1994). Another study introduces a framework for teaching and learning that integrates representations as instructional and assessment tools. Based on this research and some points of this study - "there is a close connection between mathematical language, mathematical communication and mathematical representation, the first two are indispensably important elements in mathematical representational teachings; mathematical representation is both study means and assessment tools." This paper will try to build a new teaching model of mathematical representation, and describe the process of teachers' and students' mathematical representation in the teaching. The teaching model of mathematical representation includes five steps, linked together in a cyclical process. (See in figure 2) In the following, each step of the framework will be described.

Teachers analyze the context. To set learning goals, combine teaching emphases and select teaching methods, teachers should analyze mathematical content. Mathematical content knowledge cannot be separated from mathematical language by which content is presented, so teachers need to analyze mathematical language, such as definitions, terms, formulae, etc. Besides, teachers must analyze students' prior representational knowledge, and try to obtain a balance between accurate mathematical language students should learn and their receptivity.

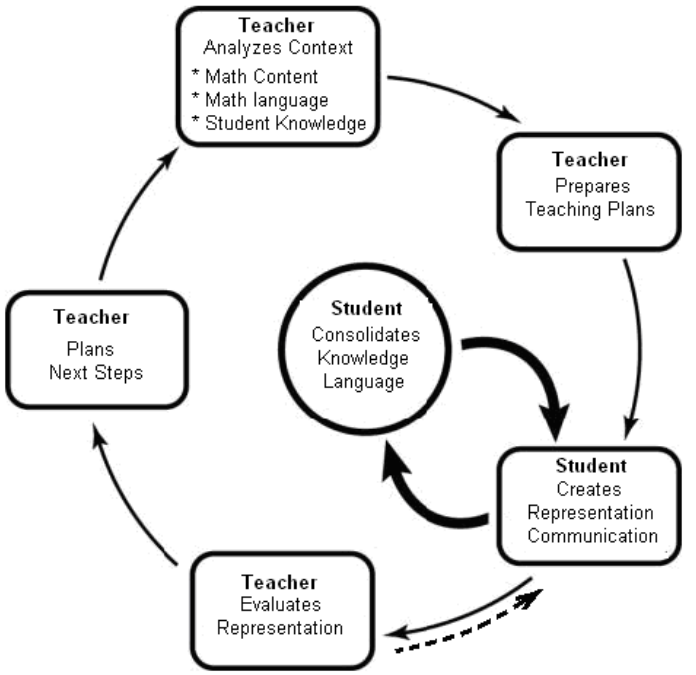

Figure 2: *The five steps of teaching model of mathematical representation.*

Teachers prepare teaching plans. Detailed context analysis is very important to effective instruction. Teachers can identify learning goals and prepare teaching plans that build on students' prior understanding to achieve a match between instruction and students' zone of proximal development.

Students create mathematical representation and communication. This step is the key of the whole framework. During the course of learning, teachers should encourage students to choose proper representations to present and communicate their mathematical ideas. Furthermore, students will be involved in a reciprocal process of using representations to communicate their ideas and develop their understanding of mathematical knowledge and language which in turn promote the mathematical representational ability.

Teachers assess mathematical representation. The fourth step is interactive with the third. Teachers analyze the representations that students have created and make reasonable assessments about the degree to which students understand what they are learning about, what learning gaps they may have, when students are achieving their learning goals, and when they are not. Then according to assessment results, teachers provide students with proper guidance.

Teachers plan next steps. The assessment results teachers draw from students' representation can provide the evidence on which to plan the next instructional steps. Depending on these assessment results, teachers decide whether to move on to new content or develop teaching content, whether to try a different teaching approach, whether to go backwards and spend more

time helping students study.

The five steps link together in a cyclical framework in which students' mathematical representation plays a dominant role, assessment is integrated into teaching, and mathematical language and communication are the two important components. Compared with the reference (Zhang, 1994) which gives a suitable teaching method for problem solving instruction, this paper suggests that teachers should use different instructional strategies for different content such as problem solving, concepts, etc. So, in terms of instructional content, this teaching model has more universal application. In fact, each step of the framework acts as a platform to extend teachers' teaching expertise. This framework is not so much a teaching model as a guide for teaching direction. It serves as a guide for teachers to integrate mathematical representational teaching into each lesson and to guide students to "learn from speaking out".

Teaching Recommendations about Primary Mathematical Representation

Teachers should seriously study mathematical teaching language. During the mathematical representational teaching, teachers need to use oral or written representation of mathematical language, and make reasonable conversion between different types of language representation. In the meanwhile, they need to guide students to represent each mathematical concept, theorem and problem with correct mathematical language, respond in a timely way to their representation to help students really understand mathematical knowledge and language. This will require teachers to seriously study mathematical language and mathematical teaching language. Teacher Xian Chu Zhao once said, "Many mathematics teachers blame that students do not study Chinese well, and Chinese teachers do not teach Chinese well, so students cannot succeed in learning mathematics. In fact, it is unreasonable to blame students and Chinese teachers. Chinese teachers only teach general Chinese knowledge; they cannot teach the words and sentences of mathematics which have their own characteristics. According to these characteristics, mathematics teachers have the duty to teach some Chinese knowledge" (Chen, 1999).

Teachers can create a "Mathematics Corner" in which students can be provided with plenty of opportunities to represent and communicate mathematical ideas. Determined by the students' characteristic and general law of representation, students will be willing to and good at representation once they are in a harmonious, democratic, united classroom atmosphere (He, 2006). Just as the most effective way to learn English is to live in England for a period of time, and so learning and using mathematical language also needs a good communication environment. So, one effective representational teaching way is to create a "Mathematics Corner" in which students will have extra time and space to ponder upon and

process mathematical language, plenty of opportunities to represent and communicate their mathematical ideas.

Teachers should strengthen language teaching and improve students' representational ability. As the key to improve students' mathematical representational ability, oral and written representational ability of mathematical language is indispensable. We hope that students could precisely represent their ideas, but they often represent them incoherently, incompletely, which shows that the process of students' thinking is not smooth; language processing is incorrect. Therefore, in order to foster students' oral and written representational ability, some strategies teachers could adopt are: helping students understand mathematical language and their relationships, make flexible conversion between different languages, and use different types of language representation to represent the same mathematical information. During the course of training, teachers should carefully analyze the teaching content, grasp mathematical language accurately, and promote the development of students' representational ability with their precise teaching attitude.

Teachers need to recognize and deal with some contradictions. Mathematical language is very precise and abstract, and teachers not only use abstract mathematical language, but vivid teaching spoken language. Therefore, in mathematical representational teaching, teachers should deal with the contradiction between students' receptivity and correct mathematical language to be used, as well as the contradiction between the ambiguity of mathematical teaching spoken language and the accuracy of mathematical language.

Based on the investigative results of primary students' mathematical representation status in problem solving, this paper tries to construct a teaching model of primary mathematical representational teaching. Due to limited space, the feasibility and validity of this model needs to be tested in practical applications. We will go deeper into research in the future.

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