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Designing Cognitively-Based Proportional Reasoning Problems as an Application of Modern Psychological Measurement Models

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This paper aims to describe the evolutionary process of developing proportional reasoning items based on a cognitive diagnosis framework. It is part of a larger research project on cognitive diagnosis assessment in the subject area of proportional reasoning. It presents changes in the formatting and structuring of two of many items during three stages, particularly by taking into account both anticipated and observed students' mathematical thinking and problem-solving behavior in proportional reasoning.

Key words: proportional reasoning, ratio, proportion, fraction, assessment.

Research in proportional reasoning describes students' ability to think proportionally as a gradual increase in local competence: it progresses from cultivating learning adaptations of multiplicative inquiries to extending transferrable mastery to proportional reasoning contexts (Cramer & Post, 1993; Karplus & Karplus, 1972; Lamon, 2007; Pitkethly & Hunting, 1996; Tourniaire & Pulos, 1985; Vernaugh, 1983). As such, students' proportional reasoning problem-solving strategies develop little by little as they gain more and more exposure to solving formal mathematical expressions of a proportion, e.g., $\frac{a}{b} = \frac{c}{d}$.

Young children begin approaching proportional reasoning problems with integer multiples using such elementary strategies as the building-up/down strategy, the abbreviated building-up/down strategy, the formal division strategy, the early-adjustment strategy, the late-adjustment strategy, the norming strategy, the unit-factor strategy, and the factor-of-change strategy (Kaput & West, 1994; Lamon, 2007; Misailidou & Williams, 2003; Tourniaire & Pulos, 1985). After realizing that these strategies may not be effective for solving proportional reasoning problems with non-integer multiples, children may look for an algorithm that is straightforward and applicable to more general proportional reasoning problems. Successful problem-solvers will eventually conclude that the cross-multiplication strategy provides an answer to this goal.

However, many researchers in the mathematics education community maintain that mastery of the cross-multiplication strategy does not necessarily equate with the ability to think proportionally (Behr et al., 1992; Carpenter, Fennema, & Romberg, 1993; Lamon, 2007; Tourniaire & Pulos, 1985). Some "successful" problem-solvers often fall into error patterns. Some of their misconceptions include the inability to differentiate between situations where proportional reasoning is appropriate and where it is not, the tendency to mismatch numerator and denominator between ratios, and the confusion to apply additive reasoning as a fall-back strategy. The practice of teaching to the test particularly exacerbates learning blind applications of the cross-multiplication strategy, which in turn leads to excessive emphasis on procedural algorithms without adequate conceptual foundations of proportional reasoning (Lamon, 2007; Lesh, Post, & Behr, 1988).

Assessments based on traditional psychometric frameworks (i.e., classical test theory, unidimensional item response theory) are generally geared toward measuring overall student performance and providing summative scores. By design, they are primarily useful in determining students' relative rankings along a continuum, and not in identifying their specific strengths and weaknesses. In contrast, cognitive diagnosis model (CDM) is a newly developed psychometric framework that can support student profiles and can be used toward formative assessment. These profiles can identify the specific fine-grained attributes (e.g., skills, cognitive processes) in which students need additional improvements. By developing attributes that are both descriptive and prescriptive, CDMs can provide instruction-relevant feedback that can encourage teachers to integrate carefully all necessary mathematics concepts in their lesson plans, including the conceptual and procedural aspects of proportional reasoning problem-solving.

Nonetheless, such psychometric applications of CDMs will only be suboptimal, or even counterproductive, if the current state of proportional reasoning problems reveals little variation in meaningful and relevant combinations of defensible attributes. As Tjoe and de la Torre (2012) show, most proportional reasoning problems commonly found in mathematics textbooks, national examinations or international assessments can be classified as missing value problems (MVPs). Consequently, designing cognitively-based proportional reasoning items necessitates starting with a validated set of attributes, such as one proposed by de la Torre and colleagues (2010): 1) prerequisite skills required in proportional reasoning; 2a) comparing and 2b) ordering fractions; 3a) constructing ratios or 3b) constructing proportions from a situation; 4) identifying a multiplicative relationship between sets of quantities; 5) determining whether two relations form a proportion; and 6) applying algorithms to solve a proportional reasoning problem.

Furthermore, if one wants to develop "good" multiple-choice format proportional reasoning items, distracters in anticipation of the likelihood of students' common misconceptions can be considered. To this end, understanding factors affecting students' problem-solving behavior in proportional reasoning is helpful. Much research on proportional reasoning shows that different structural components of proportional reasoning problems may elicit different problem-solving behaviors and thus entail different levels of difficulty (Kaput & West, 1994). These structural components of proportional reasoning problems include the location of missing value, the measure spaces of units, the dimensions of units, the partitionability of units, the coordination of measure spaces, the presence of integer multiples, and the numerical relationships involved in the problems. In addition to these structural components, students' context familiarity with proportional reasoning problems also plays an important role in influencing how effortlessly they can recognize and differentiate proportional reasoning situations.

No less important is the consideration of students' reading comprehension level. Although this skill is not incorporated in the list of attributes mentioned earlier, it is worthwhile to write proportional reasoning items at a reading level below the intended audience in middle school to ensure that students' reading ability level will not be a factor in some students' inability to solve proportional reasoning problems. Without a proactive stance, this issue might cloak an item's actual difficulty level, especially for students identified as English language learners.

Research Purpose

This paper aims to describe the evolutionary process of developing proportional reasoning items based on a cognitive diagnosis framework. It seeks to answer the research question of how to effectively design cognitively-based proportional reasoning problems in order to elicit and assess students' use of proportional reasoning attributes that the item creators intend to measure, and at the same time, to discourage students' use of other attributes that the item creators do not intend to measure. It is part of a larger research project on cognitive diagnosis assessment in the subject area of proportional reasoning. It presents changes in the formatting and structuring of two of many items during three stages, particularly by taking into account both anticipated and observed students' mathematical thinking and problem-solving behavior in proportional reasoning.

Methodology

This section describes the process of creating and modifying proportional reasoning items as part of compiling a test bank for a large-scale assessment test on proportional reasoning. The process is composed of three stages. In the first stage, four researchers and middle school mathematics teachers created and individually submitted items. In creating these original items, they specifically took into account the conceptual and theoretical issues of factors affecting student performance in proportional reasoning, as discussed in the earlier section.

In the second stage, two researchers selected 13 novel items from the collection of a total of 25 items. These 13 items were prepared to be tested out to a group of 8 middle school and 11 college students via a think-aloud protocol. Three of the 13 items involved more advanced applications of proportional reasoning in the context of college-level chemistry and physics. To make the most of the students' thinking process during the think-aloud interviews, the researchers later modified the items by changing the item format from multiple-choice to open-ended.

In the third stage, six researchers and middle school mathematics teachers collaborated to re-examine the items. They focused on incorporating into the multiple-choice distracters the difficulties and challenges that students encountered, as observed through their most common incorrect solutions during the think-aloud interviews. In addition, some items were completely changed or simply discarded. All final items were intended to be accessible at the middle school level instead of the college level. In this regard, item difficulty was another important consideration in modifying the items. Discussions of notes explaining the necessary changes of two items over these three stages were presented as follows.

Results

Item 1

In the first stage, an item creator intended Item 1 to assess students' ability to recognize and differentiate between situations where a proportional relationship is appropriate and where it is not. That is, it measured only Attribute 5. This item presented those two types of situations, and asked students whether proportion could be used to solve either one of them. In view of the literature on proportional reasoning, the item creator constructed the two easy-to-read situations by using only integer multiples and writing measure spaces in a coordinated manner. Given structural similarities in both situations, students who have mastered Attribute 5 would be expected to choose the appropriate situation. On the other hand, students would most likely choose the inappropriate situation if they practiced the cross-multiplication strategy without understanding it, failed to identify contextual differences between the two situations, and thus lacked mastery of Attribute 5. Table 1 presents Item 1 in the first, second, and third stages.

Table 1
Item 1 In The First, Second, And Third Stages

Item 1 in the First Stage	Item 1 in the Second Stage	Item 1 in the Third Stage
Look at the following two situations: Situation I: For every 1 boy, there are 2 girls in the classroom. If there are 10 boys in the classroom, how many girls are there? Situation II: Bob is 1 year old and Mary is 2 years old. When Bob is 10 years old, how old will Mary be? In which situation or situations could a proportion be used to solve the problem? A. Situation I B. Situation II C. Both Situations I and II D. Neither Situation I nor II	Look at the following two situations: Situation I: Cassandra is 10 years old and Matthew is 8 years old. When Cassandra is 23 years old, how old will Matthew be? Situation II: 5 gallons of water will fill a tank $\frac{1}{3}$ full. How much water will be required to fill the tank $\frac{3}{4}$ full? In which situation or situations could a proportion be used to solve the problem?	The proportion $\frac{1}{2}=\frac{10}{x}$ can be used to solve which of the following situations? Situation I: For every 1 boy, there are 2 girls in the classroom. If there are 10 boys in the classroom, how many girls are there? Situation II: Bob is 1 year old and Mary is 2 years old. When Bob is 10 years old, how old will Mary be? A. Situation I B. Situation II C. Both Situations I and II D. Neither Situation I nor II

In the second stage, the item creator consulted with other researchers and middle school mathematics teachers for their opinions about Item 1. A change to an open-ended format was proposed to obtain as much information about students' various thinking and approaches as possible during the think-aloud interview. After this change, the project members considered the possibility that students might attempt to perform unnecessary calculations as given in each situation. To discourage them from doing so, each situation was modified to entail more complicated numbers.

In the third stage, the researchers examined how the middle school and college students solved Item 1 during think-aloud interviews. Findings from the coding of written solutions and transcribed interviews revealed that 13% of the eight middle school students and 73% of the 11 college students provided a correct solution (de la Torre, Tjoe, & Lew, 2012). Not only did most middle school students express their unfamiliarity with the novelty of the problem type, but they also tended to commit the needless computation, thereby increasing the complexity of the problem itself. Apparently, middle school and college students answered the questions in both situations as if they were MVPs, and thus inadvertently employed unintended attributes normally associated with solving MVPs (e.g., Attributes 1, 3b, 6). This occurred even though the more complicated numbers involved in each situation were meant as deterrents to such undesired yet anticipated students' behavior.

After discussion with the middle school mathematics teachers, the researchers then modified Item 1 for the second time by changing it to a multiple-choice format. In adjusting the item to be less computationally involved and to compensate students' unfamiliarity with the novelty of the item, the original numerical characteristics of both situations were used again. Moreover, a proportion related to the numerical characteristics of both situations was included within the problem context. The question placement was also changed to encourage students to perform proportional reasoning analysis immediately after reading the first sentence of the problem. Only students' mastery to determine whether two relations formed a proportional relationship was measured in this item. That is, Item 1 assessed only Attribute 5 as originally intended. Clearly, students with no mastery of Attribute 5 would be attracted to make an improper use of the given proportion to solve Situation II, resulting in their choice of Options B and C.

Item 2

In the first stage, an item creator intended Item 2 to assess students' ability to construct an appropriate proportion given a proportional relationship. That is, it measured only Attribute 3b. Because of the low difficulty level involved in a problem that simply asked students to construct a proportion, the item creator deemed it appropriate to increase the item's difficulty level by writing the measure spaces in an uncoordinated manner. At the same time, the straightforward numerical characteristics of the problem that involved only integer multiples were intended to obviate the students' need to check whether the two pairs of integers indeed formed a proportion. More specifically, to circumvent the need to recognize a pattern of proportional relationship in the given situation (i.e., Attribute 5), the wording "a proportion" was included in the question. Table 2 presents Item 2 in the first, second, and third stages.

In the second stage, the item creator discussed Item 2 with other researchers and middle school mathematics teachers, who suggested modifying the problem's background to ease students' contextual familiarity. In particular, the word "adults" in the context of a field trip was used in place of the word "teaching assistants" in the context of a student-teacher ratio. The phrase "correctly determined" was used to provide greater reinforcement of the already established proportionality in the given situation. The item format was changed to open-ended to examine unprompted students' problem-solving approaches during the think-aloud interview.

In the third stage, the researchers analyzed the findings from the think-aloud interviews of 19 middle school and college students as they solved Item 2. Only 38% of the eight middle school students and 36% of the 11 college students correctly understood and answered what the question required (de la Torre, Tjoe, & Lew, 2012). Despite the clear indication of proportionality, many students still misinterpreted the item as asking them to check whether the given situation formed a proportional relationship, instead of simply constructing a proportion given the proportional relationship. This claim was evident in the presence of unintended attributes (e.g., Attributes 1, 3b, 6). As a result, Item 2 became needlessly more complicated than was intended to be.

Table 2

Item 2 in the First, Second, and Third Stages

Item 2 in the First Stage	Item 2 in the Second Stage	Item 2 in the Third Stage
A preschool principal made a rule that required 3 teaching assistants for a class of 7 students. Mary figured out that since there are 35 students in the preschool, 15 teaching assistants are needed. Write a proportion that could be used to show this relationship. A. $\frac{3}{7}=\frac{35}{15}$ C. $\frac{7}{15}=\frac{35}{3}$ B. $\frac{3}{7}=\frac{15}{35}$ D. $\frac{7}{3}=\frac{15}{35}$	A preschool principal made a field trip rule that required 3 adults for every 7 students. Mary correctly determined that 35 students would require 15 adults. Write a proportion that could be used to show this proportional relationship.	Determine an appropriate proportion that can be used to solve a situation below: Situation I: If 2 coins are worth 4 chips, how many chips are 6 coins worth? Situation II: In a classroom, there are 3 rows of chairs and 6 chairs in each row. If the same chairs are rearranged to 9 rows of chairs, how many chairs are in each row? Situation III: If 4 yo-yo strings are as long as 8 trumpet tubes, how many yo-yo strings are as long as 12 trumpet tubes? Situation IV: If 5 male gorillas are found for every 10 female gorillas in George's Jungle, how many male gorillas are found among the 15 gorillas in George's Jungle? A. $\frac{2}{4}=\frac{x}{6}$ C. $\frac{4}{8}=\frac{x}{12}$ B. $\frac{3}{6}=\frac{x}{9}$ D. $\frac{5}{10}=\frac{x}{15}$

The researchers raised this issue at their meeting with the middle school mathematics teachers. After several modifications, Item 2 was determined to involve an aspect of recognizing proportional relationships (i.e., Attribute 5), in addition to the elementary skill of constructing a proportion (i.e., Attribute 3b). Four situations of similar integer multiples and measure space coordination yet different contexts were included. The researchers considered Item 2 as a means of not only assessing students' proficiency in recognizing proportional relationships, but also of differentiating between students who were accustomed to the blind routines of the cross-multiplication strategy and student who had a good command of the conceptual understanding of proportional reasoning. Given such a major modification, Item 2 became more complex yet was still deemed doable at the middle school level.

The researchers chose carefully the four situations and respective options to incorporate students' most common errors. Option A would most likely be chosen by students who paid little attention to the uncoordinated order of the measure spaces, and thus incorrectly constructed a proportion based on the order in which the numerical characteristics were presented in Situation I. Option B would most possibly be chosen by students who mistakenly identified an inverse proportional relationship in Situation II as a direct proportional relationship, and thus incorrectly constructed a proportion using a constant ratio, instead of a constant product. Option D would most probably be chosen by students who lacked the ability to take notice of the measure spaces involved in Situation IV. Option C provided the appropriate proportion for the proportional relationship in Situation III in which the measure spaces were written in a coordinated manner.

Discussion and Conclusions

The evolution of the two proportional reasoning items discussed in this paper demonstrates the importance of accounting for certain structural and contextual components of problems that potentially influence different problem-solving behaviors. It became clear after a number of revisions and discussions among the item creators that in order to effectively design cognitively-based proportional reasoning problems, one should begin with an idea of the meaningful combination of attributes that are to be measured. Only then could one proceed with novel problem posing that incorporated structural and contextual components of proportional reasoning problems relevant to the specified attributes. Item creators' attributes specification, in turn, should be substantiated, or possibly modified, by evidences of students' actual use of associated cognitive processes.

An equally important aspect of item development was the presence of distracters in multiple-choice items. By accounting for students' common misconception in proportional reasoning, item creators could use distracters to reveal students' learning stages and accordingly utilize their incorrect choices to assess and develop tailored lesson plans. For example, students who consistently resort to the use of additive reasoning strategies in proportional reasoning problems can be directed towards learning gradual transition from additive reasoning to multiplicative reasoning, while students who consistently apply proportional reasoning strategies to non-proportional reasoning situations can be instructed with the difference between situations where proportional reasoning is appropriate and those where it is not appropriate.

Many of the proportional reasoning items developed in the current research project are part of a larger study on the cognitive diagnosis assessment. They are not typical proportional reasoning problems found in middle school mathematics textbooks, as mentioned earlier. For this reason, mathematics teachers may be advised that proportional reasoning items involving complex combinations of more than one attribute are more effective for a formative assessment purpose towards the end of proportional reasoning lessons rather than for an instructive purpose at the early stage of students' initial learning experiences with proportional reasoning.

From a pedagogical standpoint, these proportional reasoning items may be more constructive if they are presented to students in an organized manner. In particular, students may initially be exposed to proportional reasoning items measuring only one attribute (i.e., singleton items). Student learning may begin with a sequence of singleton items involving constructing ratios, comparing and ordering fractions, determining proportional relationship, constructing proportions, and identifying multiplicative relationship. Without a doubt, CDMs are compatible for performing profiles of students' strengths and weaknesses even for a test using only singleton items. Nonetheless, to maximize the full features of such a newly developed psychometric framework and to represent more realistic problem-solving situations, a test comprised of fewer items yet a greater meaningful combination of attributes can be administered. Hence, following intensive classroom exercise on singleton items, students may be exposed to problem-solving that involves two or more cognitive processes and eventually to applying more advanced proportional reasoning algorithms. Given a set of defensible fine-grained attributes instilled in appropriate lesson plans, cognitively-based items can serve as effective measures not only to provide informative and prescriptive assessment, but also to facilitate productive and creative mathematical problem solving experiences.

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References

Behr, M., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). New York, NY: Macmillan.

Carpenter, T., Fennema, E., & Romberg, T. (1993). *Rational numbers: An integration of research*. Hillsdale, NJ: Erlbaum.

Cramer, K., & Post, T. (1993). Connecting research to teaching: Proportional reasoning. *Mathematics Teacher*, *86*, 404-407.

de la Torre, J. (2009). A cognitive diagnosis model for cognitively-based multiple-choice options. *Applied Psychological Measurement*, *33*, 163-183.

de la Torre, J., Lam, D., Rhoads, K., & Tjoe, H. (2010, April). *Measuring grade 8 proportional reasoning: The process of attribute identification and task development and validation.* Paper presented at the Annual Meeting of the American Educational Research Association, Denver, CO.

de la Torre, J., & Lee, Y. S. (2010). A note on the invariance of the DINA model parameters. *Journal of Educational Measurement*, *47*, 115-127.

de la Torre, J., Tjoe, H., & Lew, K. (2012, April). The validation process of proportional reasoning attributes. Paper presented at the National Council of Teachers of Mathematics Research Presession, Philadelphia, PA.

Kaput, J., & West, M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 235-287). Albany, NY: State University of New York Press.

Karplus, R., & Karplus, E. (1972). Intellectual development beyond elementary school III: Ratio, a longitudinal study. *School Science and Mathematics*, *72*, 735-742.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629-668). Charlotte, NC: Information Age.

Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 93-118). Reston, VA: NCTM.

Misailidou, C., & Williams, J. (2003). Diagnostic assessment of children's proportional reasoning. *Journal of Mathematical Behavior*, *22*, 335-368.

Pitkethly, A., & Hunting, R. (1996). A review of recent research in the area of initial fraction concepts. *Educational Studies in Mathematics*, *30*, 5-38.

Tjoe, H., & de la Torre, J. (2012). Proportional reasoning problems: Current state and a possible future direction. In S. J. Cho (Ed.), *Proceedings of the 12th International Congress on Mathematics Education* (pp. 6741-6750). Seoul, Korea: ICME.

Tourniaire, F., & Pulos, S. (1985). Proportional reasoning: A review of the literature. *Educational Studies in Mathematics*, *16*, 181-204.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes*, (pp. 127-174). New York, NY: Academic Press.

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