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Day Number Routine: An Opportunity to Understand Students' Uses of Numbers and Operations

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This paper presents a "Day Number" routine involving the four arithmetic operations with multi-digit numbers. Students are challenged to use numbers and operations, according to their knowledge, to create a train of calculations in which the answer to one calculation is used to start the next one. The result of the last calculation needs to be the number of the day where the lesson takes place. Using a qualitative approach, we undertook an exploratory study that aims to identify the arithmetic knowledge that 3rd grade students activate when they are free to use any numbers and operations to calculate mentally in the "Day Number" routine. We focused our analysis on the numbers, operations and mental calculation strategies used by the students. Data were collected through direct observation, video recording, field notes and students' own productions. The analysis of students' work shows that most of them combined the four arithmetic operations envisioning to construct long trains. In all the four arithmetic operations, students used decomposition strategies; in addition/subtraction they also used sequential strategies, and in multiplication/division their options involved varying strategies.

Keywords: mental calculation strategies, multi-digit arithmetic, elementary school

Mental calculation strategies are "the application of known or quickly calculated number facts in combination with specific properties of the number system to find the solution of a calculation whose answer is not known" (Thompson,1999, p. 2). Mental calculation involves the use of personal strategies and calculations made with number values and not with digits, where operation properties and numerical relations are used. Thus, mental calculation is a thinking calculation, not mechanized (Brocardo & Serrazina, 2008) where some intermediate steps can be written on a paper (Buys, 2001). For example, to calculate 46×50, one student might need to write on a paper

40 × 50 = 2000 as an intermediate step, while others may use the commutative property to see 50 × 46 and calculate half of 100 × 46.

As Buys (2001) refers, mental calculation is a “skillful and flexible calculation based on known number relationships and number characteristics” (p. 121). Indeed, a striking feature of mental calculation is flexibility that allows students to adapt numbers to a specific operation. Flexibility is an essential aspect in the development of mathematical proficiency (Kilpatrick et al., 2001; National Council of Teachers of Mathematics [NCTM], 2000), which includes calculation fluency as the ability to calculate efficiently and adequately.

On the one hand, as argued by Gravemeijer and Muurling (2019), the digital society of the 21st century needs a high level of mathematical understanding and this is associated with the development of mental calculation hence it is based on number relations and operation properties. On the other hand, further research addressing the use of the four operations simultaneously is necessary, as well as further research into children’s multi-digit strategy competencies in the four operations and their interrelations (Hickendorff et al., 2019). In our study, we aim to give a little contribution in this domain, mainly to understand what kind of strategies students use with the four arithmetic operations in multi-digit numbers.

In this paper, we analyzed a daily routine at the beginning of the school day, named 'Day Number'. This routine consisted in challenging students to create individually and through mental calculation, a train of calculations with the four arithmetic operations until they reach the number of the day. As referred by Treffers and Buys (2001), this kind of work often gives students the opportunity to create mathematical expressions that they would find easy or normal, or which they consider difficult, but with creativity. Daily number talks, such as the “Day Number”, are a good way to move students from a procedural to a conceptual understanding, showing them the flexibility of numbers and how numbers can be constructed and deconstructed (O’Nan, 2003).

This was an exploratory study where we aimed to identify the arithmetic knowledge that 3rd grade students activate when they are free to use any numbers to operate with the four arithmetic operations in a number of the day routine, involving mental calculation and the construction of a train of calculations. We aimed to answer the following questions:

1. What kind of numbers and operations do students choose?
2. Do students rely exclusively on numerical number facts or use mental calculation strategies in their trains of calculations?

Mental Calculation Strategies in Multi-Digit Arithmetic

Multi-digit arithmetic involves more complex processes of calculation than single-digit arithmetic. Although possible, counting strategies are not efficient and retrieval is more difficult when multi-digit numbers are used.

From Hickendorff et al.'s (2019) perspective, multi-digit arithmetic is related to the way numbers are manipulated to achieve the required result and this can be done using several strategies. According to the authors, in order to be able to do so, students need conceptual knowledge about the place value system, to understand arithmetic operations and the role of the equal sign, as well as to know number facts. When memorizing number facts, it is important that students look for patterns and relationships, using these findings to build strategies. So, students should develop their ability to establish relationships between number facts in order to facilitate calculation (Fosnot & Dolk, 2001).

The development of mental calculation strategies is closely linked to a growing understanding of numbers and operations. According to students' understanding and experience with numbers and operations, these strategies can be used at different levels of complexity.

Based on previous research, Hickendorff et al. (2019) identified two main strategies that could be used in multi-digit arithmetic in addition/subtraction and multiplication/division: *number-based strategies* and *digit-based algorithm strategies*. Since this last one is based on procedural algorithms, operating on single-digits from right to left, and not on mental calculation, and it did not appear in the data, we do not present it here.

In the number-based strategy for addition/subtraction, the place value of the digits is respected (e.g., 43 can be split into 40 and 3). This category can be subdivided in four subcategories. In the first subcategory, sequential strategy, also called jump or N10 strategy students may, for example, calculate $325 - 229$, split the second number according to their place value and subtract sequentially each part $325 - 200 = 125$; $125 - 20 = 105$; $105 - 9 = 96$ (Buys, 2001; Hartnett, 2007; Rathgeb-Schnierer & Green, 2019; Thompson, 2009). The second subcategory, decomposition strategy, also called split or 1010 strategy, where students can split both numbers according to their place value and add them in separate parts. For example, to calculate 325+249 would be represented as $300+200=500$; 20+40=60; 5+9=14; 500+60+14=574 (Buys, 2001; Hartnett, 2007; Rathgeb-Schnierer & Green, 2019). The third, varying strategy, includes several strategies that requires the adaptation of numbers and/or operations to the calculation that needs to be done. In this subcategory we can include, for example, compensation strategies (e.g., to calculate 325 + 249; $325 + 250 = 575$; $575 - 1 = 574$) or inverse operations (Buys, 2001; Hartnett, 2007; Rathgeb-Schnierer & Green, 2019; Thompson, 2000, 2009) Finally, the fourth subcategory, column-based strategy, combines algorithm approaches. This strategy involves a structured vertical notation, and a number-based approach as it operates with numbers from left to right and not with digits. Some examples of column-based strategies in addition/subtraction are presented in Table 1. The number-based strategy for multiplication/division, Hickendorff et al. (2019) presents the same subcategories that we discussed above but with different meanings. Sequential strategies involve a change of operation, based on additive reasoning. To multiply, students use repeated addition (e.g., $6 \times 25 = 25 + 25 + 25 + 25 + 25 + 25$; $6 \times 25 = 150$) and to divide repeated subtraction (e.g., $45 \div 15$; $45 - 15 = 30$; $30 - 15 = 15$;$15 - 15 = 0$; $45 \div 15 = 3$). In decomposition strategy, the numbers are split according to their place value. In multiplication, one or both numbers can be split using the distributive property (e.g., $12\times16=10\times16+2\times16=160+32=192$), but in division we only can split the dividend (e.g., $168\div14$; $140\div14=10$; $28\div14=2$; $10+2=12$). In varying strategy, compensation strategies can be used (e.g., distributive property can be used in multiplication, such as 120×19; $120\times20=2400$; $120\times1=120$; $2400-120=2280$, or in division, $475\div25=19$; $500\div25=20$; $25\div25=1$; $20-1=19$) as well as doubles and halves (e.g., $12\times20=24\times10$) or inverse operations where, for example, to calculate $320\div80$, the student says 4 because 4×8 is equal to 32, so 4×80 is 320 (Caney & Watson, 2003; Hartnett, 2007; Thompson, 2009). The column-based strategy involves a vertical representation of the decomposition strategy in multiplication and repeated subtraction strategy in division, and “may act as a fruitful stepping stone, or even alternative, to the digit-based algorithms” (Hickendorff et al., 2019, p. 557). Some examples of column-based strategies in multiplication/division are presented in Table 1.

Another critical issue that needs to be considered in mental calculation is the relational meaning of the equal sign. According to Empson, et al. (2010), many students see the equal sign as an indicator that requires a calculation and an answer in the right side of the equal sign. This misconception leads students to assume that the missing number in this open number sentences $6+5 = \square + 7$ is the sum of 6 with 5 and not that 6+5 and $\square + 7$ are both equal to 11. Students also use the equality to represent a string of calculations (e.g., 38+45 is seen as 30+40=70+8=78+5=83) ignoring that there is no relation between the first operation and the number in the end of the string. Although the ability to understand the equal sign does not depend on students’ computational skills (Empson et al., 2010), highlighting the relational meaning of the equal sign during the discussion of mental calculation strategies helps students to develop their relational thinking.

Several empirical studies (Hickendorff et al., 2019; Rathgeb-Schnierer & Green, 2015; Rechtsteiner-Merz & Rathgeb-Schnierer, 2015; Rodrigues & Serrazina, 2019; Serrazina & Rodrigues, 2017) show that, before the introduction of the algorithms, elementary students were able to calculate mentally using a diversity of number-based strategies efficiently and flexibly. There is also evidence that, along with the development of mental calculation, students deepen their conceptual understanding of numbers and operations.

Methodology

In this paper, we analyzed a daily routine at the beginning of the school day, named 'Day Number'. This routine consists in challenging students to create individually through mental calculation, a train of calculations with the four arithmetic operations until they reach the number of the day. In this case, the number was 20, as the date was January 20. Students have 15 minutes to create a train of calculations that constitutes a sequence of numerical expressions, in which the answer to one calculation is used to start the next one. Thus, the result of the first calculation is used in the second calculation and so on, until the train ends with the number of the day. These trains were recorded on a sheet of paper. Students record their train of calculations on the paper to further remind and explain their mental calculation strategy. However, they were not encouraged to do it. After students created their trains, the teacher called some students to the blackboard to share their trains with the class and explain the mental calculation strategies used. The length of the train, the order of magnitude of the numbers and the degree of complexity of the calculations created, depended on students' numbers and operations knowledge. In this sense, this is a task that all students are able to accomplish, mobilizing their own numerical knowledge and personal mental calculation strategies. The discussion moment of the trains of calculations created by students allows sharing the diversity of mental calculation strategies used, thus contributing to the development of the mental calculation skills of all students in the class.

This study followed a qualitative approach (Bogdan & Biklen, 1994) framed within an interpretative paradigm (Erickson, 1986). It focused on the educational processes and the meanings of the participants: 22 students from a 3rd grade class (8-9 years old), in a public elementary school in Lisbon, and their teacher. The names of students have been changed to ensure confidentiality.

The teacher is a female expert teacher with a long teaching experience. According to the second author of this paper, who visited this teacher's classroom for two years, the teacher usually encourages students' autonomous work organized in pairs and orchestrates whole-class discussions. This teacher values the development of mental calculation.

In January, students still performed the operations through mental calculation and the algorithms were introduced later in that year. According to the 3rd grade Portuguese curriculum, it is expected that students would become fluent with all multiplication tables and perform division with remainder using informal methods and mental calculation when divisors and quotients are less than 10. The curriculum indicates the teaching of multiplication algorithm involving numbers up to one million in 3rd grade and the teaching of division with remainder algorithm with any number on the divisor in 4th grade.

The data were collected through observation by the second authors of this paper, complemented with field notes and videotaping of the whole-class discussion. The students' productions were also collected, allowing us to analyze the numbers and operations used by them. During the whole-class discussion, students shared their mental calculation strategies. Among the students that presented their trains of calculations on the blackboard, we chose four students, Mauro, Rosa, Joana and Madalena, to report here their strategies. These students were chosen because they verbalized their strategies while the other students verbalized the calculations without explaining strategies. All these data were analyzed and triangulated.

In order to answer the first question of this study, we did a content analysis of all students' work and a quantitative data treatment to determinate the frequency of the emergent categories related to number magnitude and operations used. To determine these frequencies, we counted the number of trains of calculations that fit in the defined categories. For example, concerning number magnitude, we considered the number with the higher magnitude observed in each train of calculation, allocating each train once to the category relative to that higher number magnitude. We illustrate this with Joaquim's train of calculation (see Figure 1) which was counted as belonging to the *Three-digit numbers* category although the majority are two-digit numbers. Regarding operations predominance, we considered that a train of calculations presents predominance of operations when both addition/subtraction or multiplication/division are used with bigger frequency than the remaining ones. We illustrate this with Mauro's train of calculation (see Figure 3) which was counted as belonging to the *Addition/subtraction* category since these two operations were used three times and multiplication/division were only used twice.

Inspired by previous research (Buys, 2001; Hartnett, 2007; Hickendorff et al., 2019; Rathgeb-Schnierer & Green, 2019; Thompson, 2009) we used the categorization, presented in Table 1, to analyze the second question.

Results

The Numbers and Operations Used by Students

In this "Day Number" routine, students were free to choose the first number of their train of calculation to get the number 20 (the number of the day). This routine was a way to challenge students to develop their mental calculation skills. Although the teacher did not encourage them to do so, some students saw it like a competition.

For the first number, most of the students chose multi-digit numbers multiples of 10. Only Rosa chose a single-digit number (6×6). All the others preferred larger numbers. Half of the class produced only one train of calculations and the other half produced from 2 to 5 trains. All students created operations with natural numbers and their magnitude varied between two digits and six digits. In Table 2, we present the distribution of the number magnitudes, through the absolute frequencies and percentages of the categories defined in the total of 41 trains of calculations.

This quantitative analysis shows that most of the students were fluent in calculating with numbers with more than two digits. There was a tendency (49%) to operate with three-digit numbers.

Table 1
Analytic Categories

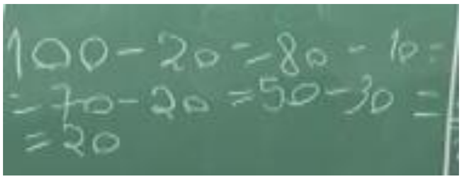
	Number facts	Number-based strategies				Digit-based algorithm strategies
		Sequential	Decomposition	Varying	Column-based	
Addition e.g., 68+24	For example, Know multiples of 2, 5 or 10 or know some sums and differences	68+20=88 88+4=**92**	60 + 20 = 80 8 + 4 = 12 80 + 12 = **92**	For example, compensation 70 + 24 = 94 94–2 = **92**	68 12+ 80 12+ **92**	1 68 24+ **92**
Subtraction e.g., 73-49		73 – 40 = 33 33 – 9 = **24**	70 – 40 = 30 3 – 9 = −6 30 –6 = **24**	For example, compensation 73 – 50 = 23 23 + 1 = **24** Or Inverse operation 49+1=50 50+23=73 23+1=**24**	73 49− 30 −6 **24**	6 13 73 49− **24**
Multiplication e.g., 13×19	For example, Know time tables	13 + 13 + 13 + … + 13 = **247** *or* 5 × 13 = 65 4 × 13 = 52 65 + 65 + 65 + 52 = **247**	13 × 10 = 130 13 × 9 = 117 130 + 117 = **247** *or* 10 × 10 = 100 3 × 10 = 30 10 × 9 = 90 3 × 9 = 27 100 + 30 + 90 + 27 = **247**	For example, compensation 13 × 20 = 260 260 – 13 = **247** Or 13 × 2=26 26×10 = 260 260-13=**247**	13 19× 100 30 90 27+ **247**	13 19× 117 130+ **247**
Division e.g., 52÷ 4	For example, Know halves	52 – 4 = 48 48 – 4 = 44 [subtracting 4s 13 times →**13**] *or* 52 – 40(**10**×4) = 12 12 – 12 (**3**×4) = 0 10 + 3 = **13**	50÷ 4= 12.5 2 ÷ 4= 0.5 12.5 + 0.5 = **13**	For example, compensation 60 ÷ 4= 15 8 ÷ 4= 2 15 – 2 = **13** *Or* *Using half of a half* 52 ÷ 2=26 26 ÷ 2=**13**	52 ÷ 4= 40 - (10×4) 12 12 - (3×4) 0	4 / 52 \ 13 4 – 12 12 – 0

Table 2
Number Magnitudes Presented in Students' Trains

Number magnitude	Absolute frequency	Percentage
Two-digit numbers	7	17%
Three-digit numbers	20	49%
Four-digit numbers	8	20%
Five-digit numbers	3	7%
Six-digit numbers	3	7%

Among all the students' work, Joaquim (see Figure 1) used only subtraction, V for used subtraction, multiplication and division, and the rest of the students (20) used the four arithmetic operations.

Figure 1
Joaquim's Train Presented on the Blackboard



As we can see in Figure 1, Joaquim did a string of calculations. He did not separate the several subtractions and so, he ignored the relational meaning of the equal sign.

Concerning the extension of each train of calculations, they varied between the use of 4 and 26 operations. In Table 3, we present the distribution of the number of operations observed in each train, through the absolute frequencies and percentages of the classes defined in the total of the 41 trains of calculations.

As we can see, the majority of the students seek to create long trains (71% create trains of calculations with six or more operations), and this has implications in several aspects, as we will discuss.

Table 3
Number of Operations Presented in Students' Trains

Number of operations	Absolute frequency	Percentage
4 or 5	12	29%
6 to 10	23	56%
11 to 14	5	12%
26	1	3%

Regarding the operations students preferred to use, we observed their predominance in each train. In Table 4, we present the distribution of the operations predominance, through absolute frequencies and percentages of the categories defined in the total of 41 trains of calculations.

Table 4

Operations Predominance in Students' Trains

Operations predominance	Absolute frequency	Percentage
Addition/subtraction	16	39%
Multiplication/division	19	46%
Without predominance	6	15%

The frequency of operations predominance shows a balance between the use of addition/subtraction and the use of multiplication/division, with more trains presenting more multiplication and division operations than addition and subtraction.

Most of the students (19) that used the four arithmetic operations started their train with multiplication or division and ended with addition or subtraction since these operations allows them to get the required 20 (number of the day) when they decided to end the train.

Table 5 presents the kind of operations used with more frequency among all the trains.

Table 5

Kind of Operations Used by Students in Their Trains

Operations	
Addition	Additions without regrouping predominating adding multiples of 10 (e.g., 11100+25) Adding equal groups (e.g., 44+44)
Subtraction	Subtractions without regrouping, predominating subtracting an explicit part of the additive (e.g., 1266−1055; 11125−1025)
Multiplication	Multiplying a multi-digit number by single-digit number (2; 3; 4; 5; 6; 8; 9) Multiplying a multi-digit number by a multiple of 10 (e.g., 195×100)
Division	Halving Dividing a multi-digit number by single-digit number (3; 4; 5) Dividing by multiples of 10 (e.g., 1000÷10; 240÷40)

Only two students, Mauro and Rosa, used addition and subtraction with regrouping, presenting a train of calculations with more complexity than the others. Only one student, Tiago, multiplied two two-digit numbers (15×15). Therefore, it seems that students create trains using calculations with which they feel safe, avoiding mistakes associated to complex computations, seeking to achieve long trains. So, they used mainly halving or halving twice (to divide by 4) and multiplications by a single-digit number and simple additions and subtractions.

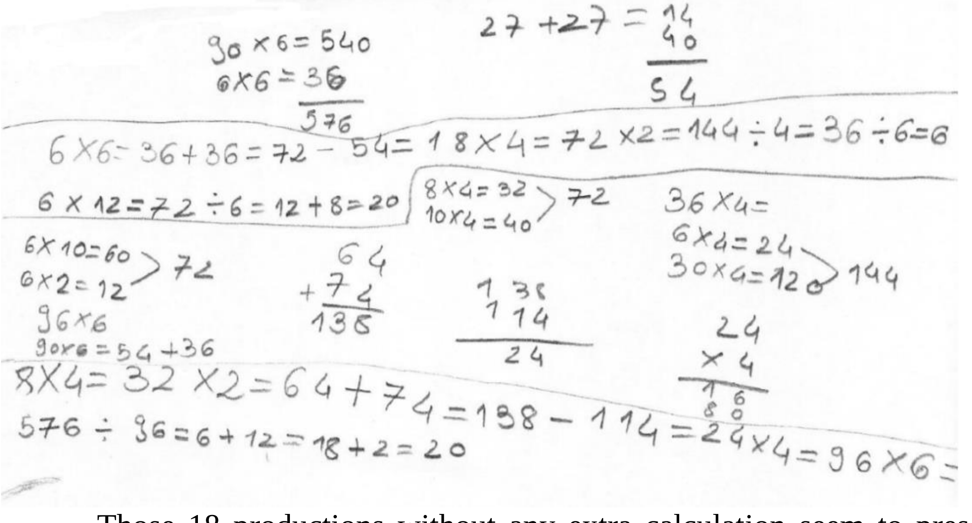
The Exclusive Use of Number Facts or Mental Calculation Strategies

Analyzing the 22 students' productions, 18 of them presented their trains without any extra calculation, as illustrated in Figure 3. Two students presented records of decomposition strategy, and two others presented a mixed of decomposition strategy and algorithms notation. Figure 2 shows the two trains Rosa created on the paper sheet, illustrative of a mixed approach.

Rosa used the decomposition strategy in some calculations (96×6;18×4; 36×4), but also the algorithms to calculate, for example, 64+74, 24×4. In the decomposition strategy she split the two-digit factor according to the structure of the number (for example, in 96×6, 96=90+6) and multiply each part by the other factor (90×6; 6×6). Then she adds the partial products (540+36=676). In this case, the strategy applies the distributive property.

Figure 2

Rosa's Trains on the Paper Sheet



Those 18 productions without any extra calculation seem to present only number facts. Therefore, it appears that most of the students chose calculations with which they feel safe to use. Nevertheless, students used strategies mentally without recording them like Rosa and Mauro did. This is not surprising since they were not instructed to record their reasoning. They only explained their reasoning when questioned by the teacher during the presentation of their train of calculations. In the case of Rosa's train, presented on the blackboard (see Figure 4), she recorded several strategies on the paper sheet, but others were done mentally and explained orally.

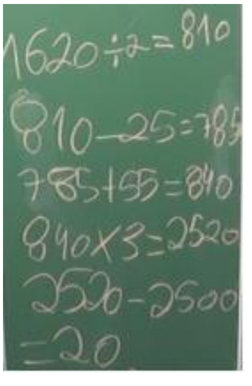
Next, we present some students' presentations on the blackboard in order to evidence their strategies.

Mauro. To get 20, Mauro started with 1620 and divided it by 2 (see Figure 3). He explained his strategies when sharing them on the blackboard:

> Mauro: I did, 1620 divided by 2 which was 810 because 1000 divided by 2 was 500 and 600 divided by 2 that was 300 and 20 divided by 2 that was 10 and I added 500 plus 300 plus 20 [not] plus 10 which gave 810. Then [to calculate 810−25] I took 10 from 810 and I get 800. Then I took 15 [out of 800] and I get 785. (…) Then [to calculate 785+55] I add 15 more, which was 800 and then I added 40 which was 840. Then I did, [to calculate 840x3] 40 times the 3 which was 120 (…) and then I did 800 times 3 which was 2400. [To calculate 2520−2500] I took 2000 [from 2520] that is 520 and then I took the 500 to get 20.

Figure 3

Mauro's Train

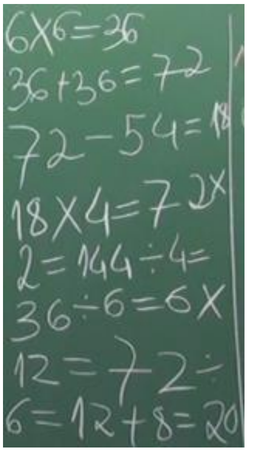


To calculate half of 1620, Mauro decomposed the dividend into 1000+600+20 (respecting the place value), divided each number by two ("1000 divided by 2 was 500 and 600 divided by 2 that was 300 and 20 divided by 2 that was 10") and added all the partial quotients to get 810 ("I added 500 plus 300 plus 20 [not] plus 10 which gave 810").

Next, Mauro recorded several operations where he used decomposition strategies to get 20. In addition and subtraction, he decomposed the second number not according to the decimal structure, but according to the numbers he wanted to use in his calculation, in order to obtain multiples of 10 first (e.g., 810−25=810−10−15 or 785+55=785+15+40) and then, added or subtracted in a sequential way. To multiply, he used a decomposition strategy (e.g., 840×3=40×3+800×3), applying the distributive property. In the last operation of the train (2520−2500=2520−2000−500) the decomposition was made according to the decimal structure of the numbers. This reveals that Mauro has flexibility and capacity to manipulate numbers to get a certain number.

<u>Rosa</u>. Rosa presented only one of the train of calculations she created on the paper sheet (Figure 4).

Figure 4
The Train of Calculations Rosa Presented on the Blackboard



In Rosas's train, the last operations are presented without repeating the result of the previous calculation in the beginning of the next one. So, Rosa, as well as Joaquim, seems to ignore the relational meaning of the equal sign.

Rosa's calculations suggest a strong use of number facts that she used in her strategies as she explained:

> Rosa: So, I saw that 6 times 6 was equal to 36 and then I did 36 plus 36 which was equal to 72. Because 35 plus 35 was 70, so if we added 2 more it is 72. Then I made 72 minus 54 which was equal to 18.
> Teacher: How did you do rapidly the expression [72−54]?
> Rosa: So, as I couldn't do 2 minus 4 in my head, so first I took 2 out of 4 to have 70, then I took 2 out of 70 that is 58 [no]... 68. Then I took the 50 and I get 18. (…).

Rosa started her train of calculations with 6 times 6, because this was an operation that she knew well, and doubled the result to get 72 based on the knowledge of the double of 35. Then she used the compensation strategy adding two more. She used a sequential strategy to solve 72−54. She split 54 into 50+2+2 and then took 2 out of 72 and got 70, then took out 2 again to get 68 and finally she subtracted 50. The discussion continued:

> Teacher: And next?
>
> Rosa: 18 times 4 that gave me 72.
>
> Teacher: How did you do that without decomposition? I don't see the decomposition there.
>
> Rosa: I decomposed but I didn't write it because I didn't have much space for that.
>
> Teacher: Ahh, then how is it?
>
> Rosa: Then I did 8 times 4 is 32. 10 times 4 is 40. (…) After, I did 72.

To calculate 18×4, Rosa used the decomposition strategy, applying the distributive property. As recorded on the paper sheet (see Figure 2), she decomposed the number 18 into 10+8 and multiplied both by 4, probably because multiplying 10 and 8 by 4 were numerical facts for her. This strategy was emphasized by the teacher, showing how she valorized it ("How did you do that without decomposition? I don't see the decomposition there.").

The teacher also drew Rosa's attention to the way she wrote the equality 18×4=72×2=144÷4, but Rosa was more focused on the operations than in the way she represented them and so she continued to explain the rest of the operations:

> Rosa: As I divided by 4 [144÷4], it was necessary to see half of 72, because I already knew half of 144. Then, I had to see half of 72 which is 36 (…) Then I did 36 divided by 6 which was equal to 6 because I had already done it at the beginning, which was 36, 6 times 6. Then 6 times 12 which was equal to 72 again. Then I did 72 divided by 6 which was equal to 12. 12 plus 8 which was equal to 20.

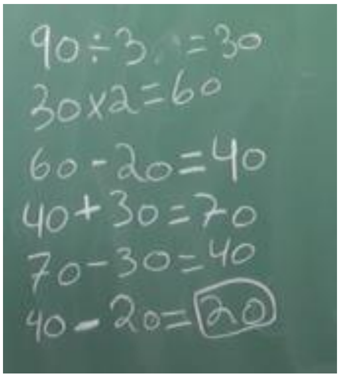
Rosa divided by 4 calculating half of a half, based on number facts that she already knew ("because I already knew half of 144"). This is a varying strategy, involving halving twice. She divided 36 by 6 recognizing the relation between inverse operations ("I had already done it at the beginning, which was 36, 6 times 6"). With the result 6, Rosa could simply add 14 to get 20 (the number of the day) but she preferred to calculate a few more operations, related with the first operations done in the beginning of the train, to get a longer train of calculations. Thus, she used the inverse operation as a varying strategy, more to get a long train than to help the calculation.

Joana. Joana's train of calculations (see Figure 5) suggests a strong use of number facts. She privileged the use of multiples of 10 since it appears that she is fluent in calculating with them.

She calculated 90÷3 thinking about the inverse operation (90÷3 and 30×3) and did it based-on number facts (3×3=9 so, 3×30=90).

> Joana: I did 90 divided by 3 which is equal to 30. Because (...) I did 3 times 3 which was equal to 9 and then I did 3 times 30 which was equal to ... 90 to divide by 3 which was equal to 30 (…) Then 30 times 2 which was equal to 60 (...). Then I did 60 minus 20 which is equal to 40. Then 40 plus 30 which was equal to 70. Then 70 minus 30 equals 40 and 40 minus 20 equals 20.

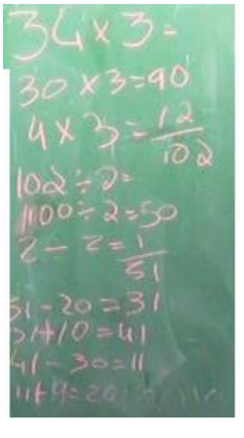
Figure 5
The Train of Calculations Joana Presented on the Blackboard



After using number facts to get 30, Joana could subtract 10 from 30 to get 20, but since the teacher asked the students to use the four arithmetic operations, she multiplied 30 by 2 (double), add and subtract numbers multiples of 10, and used the inverse operation (40+30=70; 70−30=40).

<u>Madalena</u>. Madalena decomposed 34 into 30+4 and multiplied each of them by 3 (see Figure 6), but she did not add the partial products mentally, she used a vertical notation to do it.

Figure 6
The Train of Calculations Madalena Presented on the Blackboard



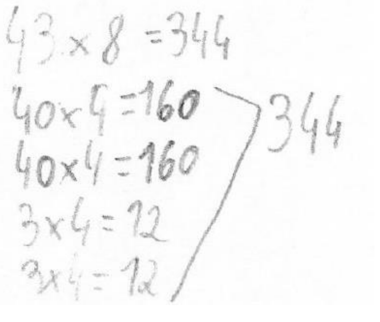
Madalena used the same procedure, as for multiplication, to calculate 102 ÷ 2 (decomposed 102 into 100+2 and divide both by 2). She used a decomposition strategy although she did the sum of partial quotients using a vertical disposition. In the rest of her train, she added or subtracted multiples of 10, except the last calculation where she added 9 to 11 to get also a multiple of 10.

All the records using decomposition strategies are linked to multiplication or division. So, it seems that this is a mechanized procedure learnt in classroom for solving multiplications or divisions. This strategy was used to calculate 43×8; 15×15; 64×8; 25×10; 34×3; 72×2; 144÷2; 18×4; 36 × 4; and 102 ÷ 2. In the case of division, the students decomposed the dividend according to the number structure. In the case of multiplication, the students decomposed just one factor (the two-digit factor), except for cases such as 43×8 and 64×8 where both factors were decomposed (the two-digit factor was decomposed in its orders and 8 decomposed in 4 plus 4). Figure 7 shows Tiago's decomposition of both factors since he was the only student that did so. Probably Tiago felt more confident multiplying by 4 than by 8.

The examples presented show that all the students were able to do this task in the time required, according to his/her number knowledge. They were free to manipulate numbers and operations to get 20, the number of the day.

Figure 7

The Decomposition Strategy of Tiago in 43×8



Discussion and Conclusions

In this exploratory study we aimed to identify the arithmetic knowledge the 3rd grade students engaged when they are free to choose numbers to operate with the four arithmetic operations in a number of the day routine that involves mental calculation and trains of calculations. We wanted to know what numbers and operations students chose and if they rely exclusively on number facts or use mental calculation strategies in their trains of calculations.

To obtain 20 (the number of the day), most of the students used multi-digit numbers (with two to six digits) and multiples of 10. Almost half of the class used a three-digit number. Concerning the number of operations used in each train of calculations by the students, 71% created trains with six or more operations with some balance between addition/subtraction or multiplication/division. The way students recorded symbolically the operations in their trains reveals that, in some cases, the equal sign is not conceptually seen in a relational way. Understanding the relational meaning of the equal sign is essential to develop relational thinking (Empson et al., 2010). The results of this study show that ignoring the relational meaning of the equal sign did not hinder the students to perform their calculations correctly. As referred by Empson et al. (2010), the understanding of the relational meaning of the equal sign does not depend on students' computational skills. Although the relational meaning of the equal sign was not the goal of this routine, being to some extent ignored by the teacher, we consider that it is a problematic issue that arises in this routine, thus requiring a special attention from the didactic point of view.

Most of the students used the four arithmetic operations (as it was asked by the teacher), but they preferred to start their trains of calculations with multiplication or division and to end it with addition or subtraction. These options seem to have allowed them to reach 20 easily at the end of the train. When adding or subtracting, students chose numbers where regrouping was not needed. They added multiples of 10 or equal groups and subtracted using part of the additive. When multiplying or dividing, they multiplied a multi-digit by a single-digit number and used the knowledge of multiples of 10. To divide, they had some tendency to calculate halves, and to divide a multi-digit by a single-digit number.

When students started their work, the teacher did not provide any information about the way they could illustrate their reasoning. In order to construct their trains, some students only recorded on their paper sheet extra calculations that could illustrate how they computed some of the results. For some students, this knowledge manifested itself when they were explaining their reasoning on the blackboard and through some students' records (e.g., Rosa and Mauro). Most of the students used decomposition strategies with the four arithmetic operations. It seems that the use of decomposition strategies in multiplication and division is something that is emphasized by the teacher, as we can see during the discussion of Rosa's strategy, where the teacher asked Rosa for the decomposition. Some of them decomposed the numbers according to their place value, showing some knowledge about place value system (Hickendorff et al., 2019) and others according to the numbers they wanted to operate (as happened with Mauro when he split 25 into 10 and 15 (in 810−25) or 55 into 15 and 40 (in 785+55) in order to obtain a multiple of 10 first). This reveals some flexibility in manipulating numbers and operations to reach a given number. Sequential strategies were also used, **e**specially with addition and subtraction and varying strategies with division (e.g., halving twice to divide by 4 or the use of multiplication to solve a division).

As argued by several authors (Gravemeijer & Bruin-Muurling 2019; Hickendorff et al., 2019), the fluency in manipulating numbers and operations in mental calculation is closely linked to mathematical understanding which favors the development of the establishment of relationships between numbers and operations.

This exploratory study allows us to perceive the importance of working on calculation routines in the classroom involving the discussion of mental calculation strategies used by students. These routines allow systematic and continuous work to enhance students' development of mental calculation. As referred by Treffers and Buys (2001), teachers must "make sure that the students often get the chance to make up problems they find easy or normal, or which they consider very difficult" (p. 73). Thus, this study showed that students who have the opportunity to create their own train of calculations, tend to rely on number facts that they master fluently and at the same time some of them feel challenged to create long trains involving big numbers and non-immediate calculations requiring the use of mental calculation strategies.

The construction by students of trains of calculations is associated with the space of creativity, an important skill for the 21st century, since students are free to create trains that, on the one hand, correspond to their level of numerical knowledge (and in this sense, all students feel capable of accomplishing the task in question) and, on the other hand, express the challenge of pretending to go further, either in train length or in the order of magnitude of the numbers used. For this reason, this free construction of the trains of calculations turns out to be also a mean for teachers to elicit the level of knowledge of numbers and operations of each of the students and thus direct the discussion to enhance that same knowledge. The two conditions imposed for the construction of the trains of calculations– using the four operations and ending with the number of the day – foster flexibility in the manipulation of numbers and in the use of operations relations.

From this exploratory study, additional questions related with teachers' practices deserve further research. What kind of practice do teachers need to have so that they can improve students' mental calculation strategies? How can the teachers articulate the diagnosis of numerical knowledge expressed in students' trains of calculations with the planning of more directed and structured tasks, aiming the development of mental calculation? How can the teachers encourage a more diversified use of strategies, expanding its repertoire, in order to be used according to numbers characteristics? A research focus in teachers practice in this field may provide important knowledge for future didactical approaches.

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