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Assessment of the Depth of Knowledge Acquired During the Aha! Moment Insight

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The study discusses the teaching experiment aimed at facilitation and assessment of Aha!Moments of insight in remedial algebra classes of an urban community college. The discussion of the study proceeds through the bisociation theory of Koestler (1964) and available instruments to assess the depth of knowledge (DoK) reached during the insight. The available instruments such as Bloom taxonomy revised Bloom taxonomy and Webb DoK are shown to be inadequate for such an assessment because of their static character. The Piaget-Garcia Triad (Piaget & Garcia, 1987) as the theory of schema development was identified to be an adequate assessment instrument for creative insight. It was used during the analysis stage of the teaching experiment as the dynamic tool to explicate three DoK levels of the study: mild, normal and strong.

Keywords: Aha!Moment, depth of knowledge, teaching experiment, creativity

The inquiry into the nature of Aha!Moment based upon the theory of bisociation have been initiated since the publication of Czarnocha, Baker, Dias, and Prabhu (2016). Prabhu (2016) described the coordination between remedial mathematics classroom practice and Koestler theory of creativity of Aha!Moment. The processes of classroom facilitation and assessment of the depth of knowledge (DoK) assessment of Aha!Moment is the subject of discussion.

This study discussed the pilot teaching experiment through weekly special assignments, which resulted in the facilitation of eight Aha!Moments. The DoK qualitative assessment approach with the help of Triad of Piaget and Garcia (1989) as the theory of schema development, reinforced by the Gestalt concept of restructuring (Weisberg, 1995) were presented and illustrated through the moments of insight collected during the teaching experiment. Three distinct levels of DoK assessment were identified: mild, normal and strong. The strengths and weaknesses of DoK assessment with the help of the revised Bloom's taxonomy for Aha!Moment insights were also discussed in this study

One may ask why devote the whole study to focus on the experience of Aha!Moment (Eureka moment). These moments of insight are very common, although often unrecognized expressions of human creativity, which take place unexpectedly at, practically, any level of cognitive development. Moreover, due to the often associated feeling of euphoria at the discovery brought by the insight, students of mathematics create an emotional positive bond with the subject. Creation of such a bond in a contemporary classroom of mathematics is a necessary condition to get students interested in the subject. The focus of the presented study delves into the depth of knowledge (DoK) generated through such an insight.

The assessment of DoK is important for the following reasons:

1. It gives information to both student and mentor about the degree of increased knowledge of the student.
2. To student such information provides a motivation resource; while to the teacher/mentor gives information about the effectiveness of the facilitation method used.
3. It gives information about the far reaches of student's ZPD, which can be used to design exercises probing these advanced levels of understanding.
4. In distinction to available measures of DoK, the described method allows measuring the degree of change in student understanding.

The aim of the presented discussion is to answer the research question: How to assess the cognitive depth of knowledge (DoK) reached during Aha!Moment insight?

In order to construct the rubric for DoK assessment one has to have a sufficiently large and diverse set of Aha!Moment descriptions. The collection used here is the composition of the results of the teaching experiment (below) and the review of the professional literature. The teaching experiment was conducted by one of the authors (BC) in the spring 2016 semester in the classes of intermediate and elementary algebra; its aim was to investigate the effectiveness of the facilitation methods for Aha!Moments insights.

Conceptual Framework

Koestler's Aha! Moment

Koestler (1964) defines Aha!Moment, Eureka Experience or bisociation as "the *spontaneous leap of insight, which connects two or more unconnected matrices of experience, frames of reference*" (p.45). The terms matrix and code are defined broadly and used by Koestler with a great amount of flexibility. Koestler indicates, "I use the term matrix to denote any ability, habit, or skill, any pattern of ordered behaviour governed by a code or fixed rules" (1964, p. 38). It follows that the term matrix can be applied to all coherent, logical or rule-based thought processes employed by individual learning mathematics:

> The matrix is the pattern before you, representing the ensemble of permissible moves. The code which governs the matrix…is the fixed invariable factor in a skill or habit, the matrix its variable aspect. The two words do not refer to

> different entities; they refer to different aspects of the same activity. (Koestler, 1964, p. 40)

Central for Koestler's theory is the notion of a hidden analogy, which gets revealed during the insight, which Koestler sees as the explosion of sudden likeness. In Koestler's opinion, and that of several other investigators such as Vygotsky (2004), the source of the discovery of hidden analogy lies in human imagination, "*it's created by the imagination', and once an analogy was created it's of course for everyone to see*" (p.17). Thus to establish the adequacy of Koestler's theory for a particular Aha! Moment, it is important to find the two or more not connected matrices that underlined the insight together with the discovery of the hidden analogy, which connects them. The analysis of the data that is of the descriptions of circumstances accompanying the insight proceeded in search for the bisociative framework through which insight took place, the hypothesis of the hidden analogy that helped to facilitate the insight. The second step in the analysis was to tease off the stages of the Piaget-Garcia Triad **(**PG Triad) from the mathematical description of the insight.

The discussion of the relationship between PG Triad and bisociation below is the component of our research, which shows that bisociation underlines several (if not all) learning theories such as the theory of anticipation of Tsur, theory of attention with its shifts of attention of John Mason, or the activity theory in the representation of Simon, and of course the theory of reflective abstraction.

Our conjecture is that any process of learning which is not based on mechanical reproduction of algorithms involves bisociation. Or, alternatively, that any accommodation or the growth of a schema of thinking, involves bisociation. Equally interesting is that bisociation became the basis of the new AI domain, computer creativity. What is really surprising that the three levels of DoK identified in the presented discussion bear strong similarity to three types of conceptual connections characterizing bisociation - based search engines (Berthold, 2012).

Piaget-Garcia Triad (PG Triad)

The Piaget-Garcia Triad (Piaget & Garcia, 1989) is a mechanism of thinking whose engine is the reflective abstraction. Its central structure is three developmental levels, Intra-, Inter-, Trans-; while the engine of development is the process is a reflective abstraction. Baker (2016) had shown that bisociation is the basis for reflective abstraction. In particular, he demonstrated how Koestler description of bisociation adds insight to understanding constructive ge3neralization a central component of reflective abstraction.

In the initial Intra- stage students experience actions or operations as isolated phenomena. They have difficulty coordinating these actions hence have a limited range in which they can successfully apply them; at this stage, creativity is very difficult and tends to be brought out only through dialogue with instructor or peers. In the second, Inter- stage, coordination between actions is observed. The solver projects, according to Piaget and Garcia (1989) a lower order schema into the problem situation and through coordination with the problem information, problem goal and other actions, he or she constructs a more general schema to solve the problem. In this stage, the different steps required to solve the problem are coordinated through bisociation (Koestler, 1964) of the concepts from their existing schema with other relevant schema and the problem situation yet these linked steps are not fully synthesized until the final Trans-stage of the Piaget-Garcia Triad.

Restructuring and Bloom-Webb DoK

The concept of "restructuring" (Weisberg, 1995) brought forward by Gestalt theorists will be helpful in the assessment of the Trans- stage of the Triad. Restructuring takes place, in the context of problem-solving, as the solver is going through an initial preparation phase that does not result in a successful solution strategy, a discontinuity in which the original strategy is disregarded, and a moment of insight in an illumination phase (Aha moment). The strategy that led to the insight approached the problem through a different analytical process - the result of the restructuring. However, restructuring as Gestalt uses the term does not describe the actual process of schema modification, which Koestler's notion of bisociation does.

Assessment of the Depth of Knowledge became widely discussed in the professional literature in connection with the Common Core design, which called for a thorough assessment of progress in student learning. One of the more popular measures has been Webb's Depth of Knowledge rubric, which is the third stage in the development of Bloom taxonomy whose first version appeared in 1956. The revision of that taxonomy took place in 2001 by Krathwohl (2001) and in 2002, Norman Webb created the DoK (Webb, 2002), which addressed the requirements of new curriculum, objectives, standards, and assessments. There is an unmistakable tendency in this process of transforming original noun-based approach to the taxonomy into the verb based in the revised version, and finally, in Webb's design, we have the expansion of terms to suggest very particular aspects of thinking attributed to each stage. For example, where Blooms taxonomy classifies the stage of Application as *the use of learned materials in new and concrete situations,* revised Bloom taxonomy calls it the Applying stage, while Webb's taxonomy calls it the stage of Skills and Concepts, describes it as "*Engages mental processes beyond habitual response using information or conceptual knowledge*" (Webb, 2002) and provides large collection of verbs, which, classified into four cognitive levels, do that job.

While possibly helpful to teachers in terms of assessing the state of knowledge of their students, it has important flaws as the method of DoK assessment, especially in the context of Aha!Moment creativity and insight:

1. Investigation of the depth of knowledge reached during the insight has to show the degree of progress of understanding, and that means to show the difference in understanding before and after the insight. The positioning of the DoK on a particular level of discourse does not help in this process so that the assessment of the progress is imprecise. Moreover, it does not give any information about the process that helped in the insight. Such information is helpful to the teacher in developing her/his teaching craft by addressing particular limitation or epistemological obstacles encountered by students.
2. The static character of the division into the stages does not allow analyzing student thinking, which went through several stages of development as a consequence of the insight.
3. We are suggesting that the assessment of the depth of knowledge of Aha! Moments has two dimensions, one of the Bloom-like characterization of depth, another of intensity counted in the number of steps of progress in understanding what occurred.

Methodology

Site

This study was conducted at an Urban Bilingual (Spanish/English) Community College in New York City, Bronx. Eugenio María de Hostos Community College was established in 1968 in the South Bronx an inner-city college in the City University of New York (CUNY) system to meet the higher educational needs of people who historically have been excluded from higher education. That is to provide access to higher education leading to intellectual growth and socio-economic mobility. At present, the college enrolls 6500 students. The college student profile indicates that the student body is approximate: 66% Female, 59% Hispanic, and 23% Black, with 13% not known. Approximately 85% of entering student require remedial mathematics about 59% of students have HS degrees while 15% have G.E.D and 21% have HS degrees from another country. In general, they are young adults, as well as returning students of a variety of ages.

Participants

Thirty-five students from two remedial algebra classes participated in this study. They were in majority females, usually first in the family who have gone to college. Together with students, there were three female peer leaders in these classes with more advanced mathematics knowledge acting as student/researchers

Data Collection

Descriptions of Aha!Moments were collected from students and peer/leaders. The phenomenon of Aha!Moments was discussed at the beginning and throughout the semester with both classes. Students were asked to fully describe the insight experience in terms of mathematics they were working with at the moment of insight. Students were assigned biweekly Special Problems, chose from different collections such as Internet's Open Middle platform as well as problems from high school professional development. The guiding idea of Special Problems was to choose off-track problems requiring at least a moment of thought. Some Aha!Moments were reported by peer leaders who encountered them in different classes of mathematics and physics.

Procedure

The teaching-research methodology Teaching-Research/New York City (TR/NYCity) Model has been described in several publications, among them in Czarnocha, Baker, Dias, and Prabhu (2016). It involves instructor/mathematics Ph.D. who plays a dual role, that of a teacher and that of a researcher. The instructor teaches, observes, notices, and asks leading questions and takes notes during the classroom to be discussed later with the TR Team. The TR Team of the described teaching-experiment included student-researchers present during each class, who as more experienced peers worked with students during and after the class. The TR Team met weekly for the discussion and validation of classroom observations, Aha!Moments and techniques of facilitation. In addition to facilitating classroom student moments of discovery, the student-researchers were asked to observe themselves in more advanced mathematics courses they were taking and with the description of their own moments of insight.

There were two methods of facilitation introduced during the teaching experiment: weekly/biweekly special moderately advanced problems assigned to be solved at home by each student. Correct solutions were awarded special credit, which later counted highly in the final grade of the student. The second method was scaffolding of a particular student understanding during short classroom interactions with the student.

Data Analysis

We classified strong bisociation as that one which has at least two steps and/or two cycles in progress of understanding. The next, normal level of bisociation is the building up process in which several of the elementary schemes are coordinated to form a functional whole. We classified mild bisociation as that one that involves only one mathematical step or analogy. This includes the process in which discovering a hidden analogy involves employing elementary schemes that are intuitive or self-evident to the solver and thus seen as relevant but in which the path to a solution is not entirely clear.

Results

In the sequel below, we presented the description of the some relevant Aha!Moments in its totality; of others, due to the space restrictions, only relevant fragments of the descriptions were offered.

Fir Tree Aha! Moment

A strong bisociation is especially important because it shows that an Aha! Moment does not have to occur in the context of problem solving. In this standard problem a sequence of rectangles symbolizing the fir tree is visually presented with the help of 2,6,12, and 20 unit squares for the number of the figure n= 1,2,3,4 as the diagram begins to grow (see Figure 1). The solver is asked first how many rectangles for n=10, then what is the general form of the equation as the function of “n”. One student determined an equation for this pattern as the n^{th} diagram has n (n–1) rectangles. Here, the Aha!Moment took place as further refinement of the solution, which was obtained before the insight took place through the standard process of abstraction and generalization of a pattern. Once the solver abstracted the relationship n (n – 1) as representing the value of squares at stage ‘n’ she coordinated this with the problem situation to gain further insight. She realized that the fir tree will contain N squares if there is a ‘n’ such that n(n − 1) = N and at this point reflection upon solution activity to find such an ‘n’ leads her to synthesize an understanding of how to solve this with her understanding of factoring trinomials. Again, this abstraction creates a new code or scheme in which she reduces the problem to whether one can factor the trinomial $n^2 + n - N = 0$. In this situation, the ‘building up’ process consists of the solver repeating variations perceived as related but distinct cases of solutions until they abstract or synthesize the common code.

Square root domain - a strong bisociation. The problem starts with the function $f(x) = \sqrt{x+3}$. The teacher asked the students during the review: “Can all real values of x be used for the domain of the function $\sqrt{x+3}$?”
Student (S): (1) “No, negative x’s cannot be used.”
Teacher (T): (2) “How about $x = -5$?”
S: (3) “No good.”
T: (4) “How about $x = -4$?”
S: (5) “No good either.”
T: (6) “How about $x = -3$?”
Student, after a minute of thought: (7) “It works here.”
T: (8) “How about $x = -2$?”
S: (9) “It works here too.”
A moment later, the student adds: (10) “Those x’s which are smaller than -3 can’t be used here.” T: (11) “How about $g(x) = \sqrt{x-1}$?”
Student, after a minute of thought: (12) “Smaller than 1 can’t be used.”

T: (13) "In that case, how about $h(x) = \sqrt{x - a}$?"
S: (14) "Smaller than *a* cannot be used." (*Second creative generalization*)
The bisociation takes place in line 10 when the student finds hidden analogies in the concrete examples discussed earlier during the dialogue, which are abstracted and generalized. A similar process takes place in lines 11-14.

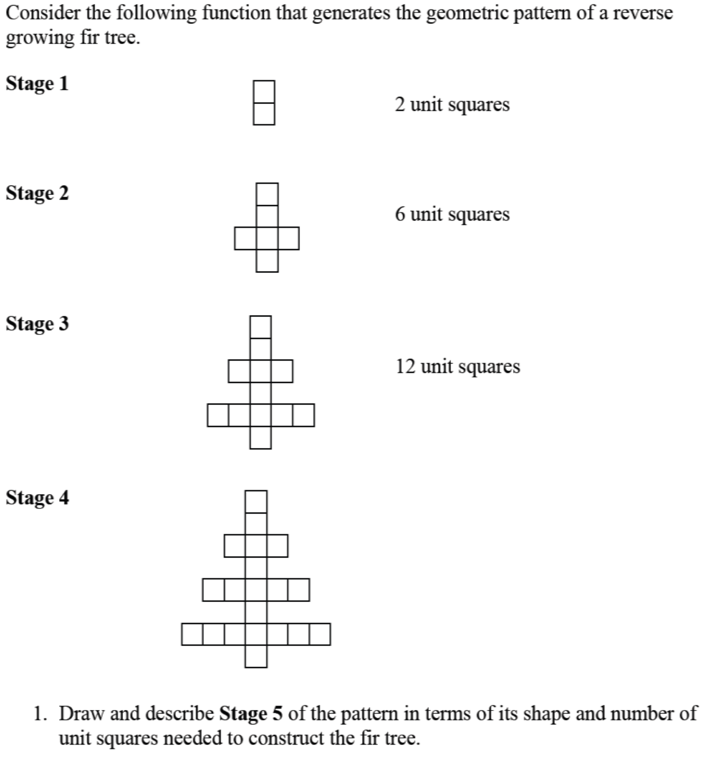

Figure 1. *Fir tree aha! Moment.*

The student makes an incomplete association from the matrix of finding the domain of a function such as f(x) = $\sqrt{x}$ to this problem situation which involves a transformation of the previous situation (the argument within the square root, x+3)

In lines (6) and (8), the instructor employs concrete counterexamples to provide a *perturbation*, or a *catalyst*, for cognitive conflict and change. "*...perturbation is one of the conditions that set the stage for cognitive change*" (Von Glasersfeld, 1989a, p. 127).

In lines (**6**) – (**9**) the student reflects upon the results of the solution activity. Through the comparison of the results (records), they abstract a pattern, **—** "*the learners' mental comparisons of the records allow for recognition of patterns*" (Simon et al., 2004).

Physics Aha! Moment, a normal bisociation. It takes place exactly from the Analysis to Synthesis level of understanding. In the words of the student:

I could not bring myself to figure this out so I began to play with the triangles within the prism to see if anything could jog my memory. After a lot of sketching and redrawing, I tried the two right angle triangles. I thought that my thought process was wrong since the two right angles that I could see were both different sizes. Using the right triangle rules [and] opening the right triangles on a piece of paper I was able to see the problem differently. When I opened the right triangles I noticed that they were the same [similar]and since we were dealing with variables and not actual numbers I could apply the given variables…I determined that the blue triangle and red triangle were equal when dealing with variables, as shown below (see Figure 2).

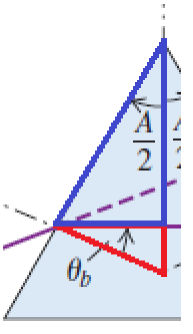

Figure 2. *Physics aha!Moment.*

The description of the circumstances which accompanied the Aha!Moment here conform with the Analysis level of Bloom described as "Break(ing) down material into component parts so that its organizational structure may be understood" followed by the Synthesis described as" Put(ing) parts together to form a given whole" and/or characterization of creativity by "Put(ing) elements together to form a coherent or functional whole…." (Revised Bloom taxonomy).

Aha! Moment to factorize Students were asked to find integer factors of $2x^2 + 3x + \square$.

We have here Normal to strong bisociation in the context of trial and error method, suggesting restructuring of the problem to integrate algebraic and visual. Therefore, we have here first Application of the coordination; the difficulty with Application was the force that propelled the student to algebraic generalization as Synthesis. Again Aha! Moment took place between the levels of Bloom taxonomy. The hidden analogy leading to Aha! Moment was between the concrete examples and the general form of factorization monomials. The student solves the problem with aide of visual rectangles finding first factoring the polynomial as: $2x^2 + 3x + 1 = (2x + 1)(x + 1)$. Seeing how it works, she had the gentle, but definitely noticeable on both cognitive as a generalization and effective as empowerment level, the Aha! Moment, which sent her to formulate the general condition of $(2x + a)(x + b)$ as the hidden analogy between examples she did. This experience empowers her and she begins to search for other factors, then after a period of time finds another solution one with a negative factor: $2x^2 + 3x – 2 = (2x – 1) (x + 2)$.

After the point where she/he chooses a negative integer, the student realizes that there is always the same pattern to this multiplication that is the code can be written as: 2a + b; the student has translated or abstracted the code she/he is using during this trial and error process into an algebraic equation a + 2b = 3. This certainly qualifies as reflective abstraction using the definition of Simon et al (2004) as a reflection as based upon goal-directed solution activity during which the solver has mental records of the result such activity. The solver's mental comparison of these records allows them to see a pattern and the abstraction of the activity-effect relationship is the beginning of concept formation a "coordination of conceptions" (Simon et al, 2004 p.319).

The Elephant, a normal bisociation. Two pupils, age 10, one listed as tutor the other as solver are working together to solve the word problem: The sum of two numbers is 76 one number is 12 more than the other find the two numbers.

The difficulty of the student is in understanding the concept of the unknown. The tutor tries to explain the concept using the metaphor of intervals, then boxes/windows and finally he chooses the figure of the elephant as the tool of explanation.

Calculus problem. The instructor gives briefly introduces the limit concept and then asks to find: $\lim_{x \to 0} \frac{\sqrt{1-x}-\sqrt{1+x}}{x}$. In this situation, the student recalls rationalizing the denominator from an earlier class (hidden analogy) and experiments with the technique in this situation.

This is a mild bisociation. The solver goes through a clear period of uncertainty that nicely fits Norton's (2008, 2009) description of abduction that is to say the solver recognizes a previous scheme that is related and conjectures that it will help solve the problem but is uncertain over the results as there is clear difference between his previous mental records or scheme and the current situation. "I was looking at the limit, and said to myself why not apply the same rule for the fraction when we have the radical in the denominator". Consequently, the hidden analogy is here between algebraic expression positioned in the numerator and denominator. Thus, the solver conjectures his existing scheme is relevant and coordinates this with the problem information to produce a new scheme.

Discussion

Two modes of facilitation, complex problems solving for individual or group work, and scaffolding of student understanding leading to conceptual learning turned out to be fruitful in the facilitation of creativity in mathematics remedial classes. The majority of Aha!Moments were recorded by the student/researchers in their more advanced classes, however several of them were observed in remedial algebra classes indicating that the level of mathematics is inessential in the process of facilitation. It is also inessential from DoK point of view. Elementary insight can be quite deep.

Fir Tree Aha Moment

Both Hershkowitz, Schwartz, and Dreyfus (2001) and our solver record variations in solution attempts in a table form, while in Simon (2014) the student abstracts the results of solving several different related problems all presented as unrelated and without the use of tables. However, we argue that in each case there was a 'building up' phase as the student worked on examples they considered separate but related and then a bisociative synthesis of the codes or structural abstraction that unified these examples and propelled the solver into the second and third stage of the Piaget triad. In terms of Bloom taxonomy, we have here the synthesis level of the pattern generalization with the concept of a quadratic equation and the role of factorization in solving the equation. The extended thinking of Webb's Depth of knowledge suggests "complex reasoning", however, no planning through an extended period of time was necessary as the Aha!Moment of bisociation is instantaneous, though solving the original problem can be taken as the required extended time.

Square Root Domain

Norton & D'Ambrosio (2008) credit Vygotsky with the view that the ability for conscious and meaningful imitation of an instructor or peer tutor comes in the form of instantaneous insight when the learner is ready i.e. when they have developed enough schema knowledge to interpret and internalize new actions. They consider a two-step process for internalization of concepts and processes understood by others but new to the learner; in the first stage, the learner requires assistance and in the second, she moves towards internalization by herself. In this first stage, the instructor's role is to use scaffolding to prepare the learner for the second stage. The difficult question at this stage mentioned by Norton and D'Ambrosio is for the instructor to asses when meaningful internalization has taken place as opposed to rote and meaningless repetition. One critical component of scaffolding lies within the instructor-learner dialogue during problem-solving and in this example, the instructor employs scaffolding technique of pointing out a student error or more accurately leading the student to see their error so they could reassess their work and move towards the internalization step characterized by bisociation of this new situation with previous knowledge.

Evidence of the abstraction of the principle that the transformation of the argument $x+3$ beneath the square root shifts the domain 3 to the left is given in line (10). Thus, this realization can be seen as a reflective abstraction in the sense of Simon, Tsur, Heinz and Kinzel (2004). In that, the solution activity of substitution was projected into and coordinated with the solver's knowledge of the domain of the square root function it is also an example of reflective abstraction according to Piaget and Garcia (1989). In that solution activity through substitution with the student's matrix or schema of evaluating x – a is coordinated or integrated by the student with the student's understanding (matrix) of the domain of the square root function their realization in line (10) can be viewed as bisociative in nature.

In lines (11) and (12), the perturbation brought about by the teacher's questions leads the student to enter the second stage of the Piaget and Garcia's Triad. Thus, the student's understanding of finding the domain of a square root has undergone constructively generalization to accommodate transformation providing evidence of the structural understanding noted by Sfard (1991) and the third stage of the Triad (Piaget & Garcia, 1989).

Analysis of DoK through Bloom's taxonomy for this case reveals the difficulty in stable assessment of the student progress. The dialog goes through the Application stage facilitated by the instructor; student jumping over the Analysis stage to land in the Synthesis as an abstraction and generalization. Then again Application of the new understanding and second Synthesis. So, the Aha! Moment is again taking place, twice through iteration in between levels of the Bloom Taxonomy.

Physics Aha!Moment

Strategic thinking of Webb's DOK stage is reflected in reasoning, differentiating and development of a plan (*opening triangles and noticing they are the same*) while the Extended Thinking stage is reflected by the student in terms of connecting different triangles through similarity, critique of the situation as well as explaining phenomena in terms of new concepts.

We have here a standard example showing that what is important in the assessment of DOK is the extent of the gap, which was dealt with during the insight. Here, the Aha! Moment or bisociation was within the transition between Analysis and Synthesis level.

Aha!Moment to Factorize

In our view, Simon's understanding of reflective abstraction moves closer to Koestler understanding of bisociation when he states that, "knowledge of the logical necessity of a particular pattern or relationship is generated through reflective abstraction. By anticipation of the logical necessity…As a result of reflective abstraction, the student learns not only that the relationship exists but why the relationship is necessary" Simon (2006, p.365). For Simon understanding, logical necessity implies the solver not only notices the pattern but also further understands the reason for the pattern i.e. the equivalence of the mental records she has accumulated and can thus anticipate her actions and their effect within this problem type. *"...she came to see the equivalence of particular problems and can explain the logical necessity of their relationship.*"

(Simon, Saldanha, McClintock, Akar, Watanabe & Zembat 2010, p.99). In the terminology of Koestler, one would say she synthesized the codes she was reflecting upon in the pattern of her experiences with her object level understanding of variables to create an artifact i.e. the algebraic expression 2a+b = 3. Once again this, like the earlier fir tree problem, can be seen as the synthesis of codes during a 'building up' in the abstraction. This synthesis takes previously related but separate solutions into a unified scheme and is thus structural abstraction within the Piaget Triad.

The Elephant

This episode represents what Vrunda Prabhu (2016) refers to as a bisociative nature of a creative learning environment when the tutor realizes that the usual symbols or artifacts that typically introduce the concept of a variable such as a visual symbols of intervals or boxes/windows were not sufficient so he instead used a concrete visual symbol of the elephants that happened to be in the room and this realization led to the other student's successful realization of the variable concept. It is an example of the Aha!Moment realization by the student and by the tutor in one mathematical situation.

The elephant Aha! Moment takes place from below the Bloom's first level to the second level of understanding. The hidden analogy is between the logic/geometric approach and perceptual matrix. The student B. has difficulty with the concept of the unknown in the context of a linear equation. The bisociation takes place when his peer, student P. realizes he has to reach beyond mathematics into the different perceptual matrix to be able to explain the concept to B. What makes a normal bisociation is the length and variety of the process that led to the single bisociation. This problem is noteworthy because it demonstrates a teaching research bisociation as there are two realizations occurring one by the learner and the other by his peer-tutor. The one student-learner finally makes the abstraction required to consider an unknown as an object with the analogy of the elephant as being some unknown quantity the other realization is by his peer tutor who realizes several times that the analogy he has presented is not being understood by the other student and thus he changes his chosen analogy from line to box to window and finally to an elephant, thus demonstrating natural teaching skills.

The Calculus Problem

This process is referred to by Norton (2008) as a type of accommodation called "generalizing assimilation" in which existing schemes are applied intact to new situations with only slight if any modification the accommodation-change being that their domain of application is extended.

Conclusions

The collection of Aha! Moments has arranged along the intensity (or depth) gradation axis of mild, normal and strong. We classified mild bisociation as that one that involves only one mathematical step or analogy, this includes the process in which discovering a hidden analogy involves employing elementary schemes that are intuitive or self-evident to the solver and thus seen as relevant but in which the path to solution is not entirely clear; the normal level of bisociation is the building up process in which several of the elementary schemes are coordinated to form a functional whole; by a strong bisociation we classified as that one which has at least two steps and/or two cycles. The second or structural abstraction often occurs during the building up process in which a single solution process built is repeated in what is understood by the solver as separate but related solution activity. Often the results recorded in a table until synthesis of codes allows the solver to understand the logical necessity behind their solution activity and thus to engage in a structural abstraction. One of the central questions for the development of Aha!Pedagogy that is pedagogy based on the facilitation of Aha!Moments is how to retain the discovered concepts and incorporate them into the student schema. Knowledge of the PG Triad level of the DoK allows designing retention exercises, which correspond to the cognitive level of the discovery. Such exercises should be given to student(s) soon after the original insight took place.

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