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Learning Rational Numbers through an Emergent Modeling Process enhanced by Percentage

Helena Gil Guerreiro
Lurdes Serrazina
UIDEF, Instituto de Educação, Universidade de Lisboa, Portugal

In this paper, a discussion about how elementary students' conceptual understanding of rational numbers is fostered, along with number sense development, in a sociocultural learning perspective, is carried out. We highlight how an emergent modeling process, enhanced by percentage and supported by multiple representations, emphasizing numerical values of magnitude contexts, may contribute to that understanding. A classroom experiment was developed, following the methodological procedures of a Design Research. We analyze student's interactions and their work, as representations are used and transformed as models. Findings suggest that percentage, associated to students' real world experiences and previous knowledge, is intuitive and seems to be helpful in students' numerical development from whole numbers to rational numbers. In a classroom community that supports a co-participated learning, where multiple representations serve and become models as dynamic process, can be a way to reach the meaning of rational number's concepts and to strength interrelationships between those concepts.

Keywords: Elementary school mathematics, co-participated learning, rational numbers, models, percentage.

In mathematics education research, learning and understanding rational numbers has been studied widely up until today and are commonly considered very important, but also of increased complexity (Behr, Lesh, Post, & Silver, 1983). In a perspective that emphasizes a crucial continuity between learning whole numbers and rational numbers, a numerical development approach based on magnitude representations and supported on the development of students' conceptual understandings can show improvements in the rational numbers' learning process (Moss & Case, 1999; Siegler, Thompson, & Schneider, 2011). This idea of continuity is consistent with number sense, a process of progressively broadening development, since early ages, of useful, flexible and efficient numerical strategies and relations (McIntosh, Reys, & Reys, 1992). Percentage, as a representation common in everyday life, which emphasizes measurements of quantities through comparative relations, can be a significant stepping-stone in rational numbers' learning process.

Within a broader study that seeks to understand how elementary school students construct their understandings of rational numbers[1], in the social context of the classroom, a classroom experiment has been conducted, regarding number sense and based on a sociocultural learning perspective. In this paper, our central question is to provide useful insights onto how an emergent modeling process supported by multiple representations and enhanced by percentage in a measurement context, may be an anchor to foster students' conceptual understandings of rational numbers.

Learning Rational Numbers with Understanding in Elementary Grades

In elementary grades, rational numbers emerge as a natural extension of the way that students use numbers, allowing to solve problems that are not possible to solve with just whole numbers (NCTM, 2010). Conceptual understanding of rational numbers can be trigged by coordinating students' intuitive and existing understandings of numbers with the development of a qualitative structure for proportional evaluation, supported by percentage (Moss & Case, 1999). By focusing on percentage, a network of concepts, schemes, and symbols is constructed out of the old ones. It presumes an integrated understanding of rational numbers, regarding the meaning of numbers and the contexts in which they are important, as stressed by Moss and Case (1999).

Deepening Conceptual Understandings of Rational Numbers

Rational numbers can be difficult to teach due to the complexities with the concepts they involve. Literature in this domain points out several explanations for the difficulties that students have to deal with, suggesting that attention should be paid to the development of students' conceptual understanding, instead of emphasizing procedural skills and computational algorithms (Fosnot & Dolk, 2002; Monteiro & Pinto, 2007; Moss & Case, 1999; NCTM, 2010). Thus, conceptual understanding is one of the strands of mathematics learning that some authors believe necessary for effective learning. It means being able to relate mathematical ideas, understanding their relationships in an integrated way into a coherent structure of concepts, operations and relations (Kilpatrick, Swafford, & Findell, 2001). It draws from Skemp's (1976) conception of relational understanding as a meaningful kind of learning that concerns the learning process, with its links and relationships, which is useful, as the author stresses, to explore ideas further.

This conceptual understanding is closely related to number sense, as it refers to general understanding of number and operations and to the ability of using it in flexible ways, in order to make mathematical judgments and develop effective strategies in dealing with numbers, and rational numbers as well, in different situations (Abrantes, Serrazina, & Oliveira, 1999; Mcintosh, Reys, & Reys, 1992). It is a process that students should gradually develop from early age. It

[1] The term *rational numbers* refers to the set of non-negative rational numbers.

requires not only to acquire flexibility in working with the various number representations, more or less conventional, with which students feel confident in moving between (Ponte & Serrazina, 2000).
Employing multiple representations, enactive, iconic, symbolic (Bruner, 1962) and oral and written language (Ponte & Serrazina, 2000), has long been an issue of considerable interest for understanding a mathematical concept, through a richer and deeper perspective. Therefore, representations are considered useful tools in the development of conceptual understanding as they help students to track ideas and inferences when reflecting and structuring a problem. Gravemeijer (1999) reinforces that a model emerges when it is underpinned by representations. In this emergent modeling process, representations serve and become models, which can support the emergence of more formal mathematical knowledge (Gravemeijer, 1999). Gradually, students shift their attention from the situation, where the model is a *model of* acting in a specific context, to the mathematical relations involved, where the *model* goes *for* a more formal reasoning (Gravemeijer, 1999). Consequently, looking at rational numbers, considering elementary grades, emergent modeling is a dynamic process required to co-develops representation and conceptual understanding.

Percentage as an Eminently Social' Interpretation
In the Portuguese educational context, percentage is not a common entry into rational numbers' learning process. Although, students become familiar with percentage in real life situations, as it is present in food labels, clothing tags and discounts. Thus, students develop an intuitive sense in dealing with percentages before being introduced to it at school (Moss & Case, 1999). Moreover, in Portuguese, the term *percent* (in Portuguese *por cento*), has a similar written form as the cardinal *one hundred* (in Portuguese *cento* or *cem*). *Percent*, meaning by the hundred, is said in Portuguese *por cem*. This phonetic similarity seems to be helpful in the way students perceive percentage in an early approach. This familiarity with percentage challenges its formal introduction in elementary education, as part of the topic of rational numbers, to foster rational numbers conceptual understanding. Percentage, with its perceptual message and multiple interpretations, can be seen as a language of proportion that has different meanings as it assumes properties of number, part whole, ratio, function or statistic (Parker & Leinhardt, 1995). Therefore, exploring percentage can be an opportunity to begin thinking relatively and to deal with multiplicative situations, enhanced on students' early intuitive understandings of proportional relations (Lamon, 2007). Learning percentage at elementary school means to take advantage of its relational language using elementary strategies, including benchmarks, proportional reasoning, and additive building-up strategies, rather involving formal calculation procedures (Moss & Case, 1999; Parker & Leinhardt, 1995). This requires the development of appropriate models to grasp rational numbers' sub constructs of measure, part-whole, ratio, operator and quotient, but also its relational features, in order to

support students' attempts to make sense of the numbers and the relationships that connect them (Dole, Cooper, Baturo & Conoplia, 1997; Parker & Leinhardt, 1995). Models associated to percentage can provide support for developing rational numbers' conceptual understanding. It can be used to display relationships between quantities or magnitudes values' in multiple ways (Parker & Leinhardt, 1995).

From a sociocultural learning perspective, mathematical understanding is taken as a product of social processes, in particular as a product of social interactions (Voigt, 1994). From this point of view, conceptual understanding must be seen as emerging between students when communicating with one another, through processes of negotiation of mathematical meanings. In this perspective, numerical development is seen as a continuous process with commonalities and differences that are part of the construction process and must be stressed and discussed in the classroom. A process of co-participated learning (Pontecorvo, Ajello, & Zucchermaglio, 2005) that happens in a relationship of reflexivity from both individual and collective mathematical activity, at classroom learning community (Cobb, Stephan, McClain, & Gravemeijer, 2001). Thus, students must be encouraged to communicate their strategies and solutions, explain their reasoning strategies using representations, namely through whole-class discussions (Wood, Cobb, & Yackel, 1993).

These theoretical elements are considered key pillars to support the interpretative framework of this research, which was inspired in Moss and Case (1999) experimental curriculum, where percentage is the cornerstone for developing students' conceptual understanding of the rational numbers.

Method

Research Design

In this study, we developed a classroom-based design research (Cobb er al., 2001) and we analyzed the collective learning as it occurs in the social context of the classroom. After analyzing the results of a preliminary study, taken with diagnostic purposes, a classroom experiment was designed and developed, involving twenty instructional tasks. It was based on a conjecture with a mathematical content dimension and a pedagogical dimension. The first dimension implies a hypothetical learning trajectory, inspired in Moss and Case´s (1999) experimental curriculum for teaching rational numbers. This trajectory can be schematized in a three stages diagram (see Figure 1), although should not be interpreted in a fixed compartmented way.

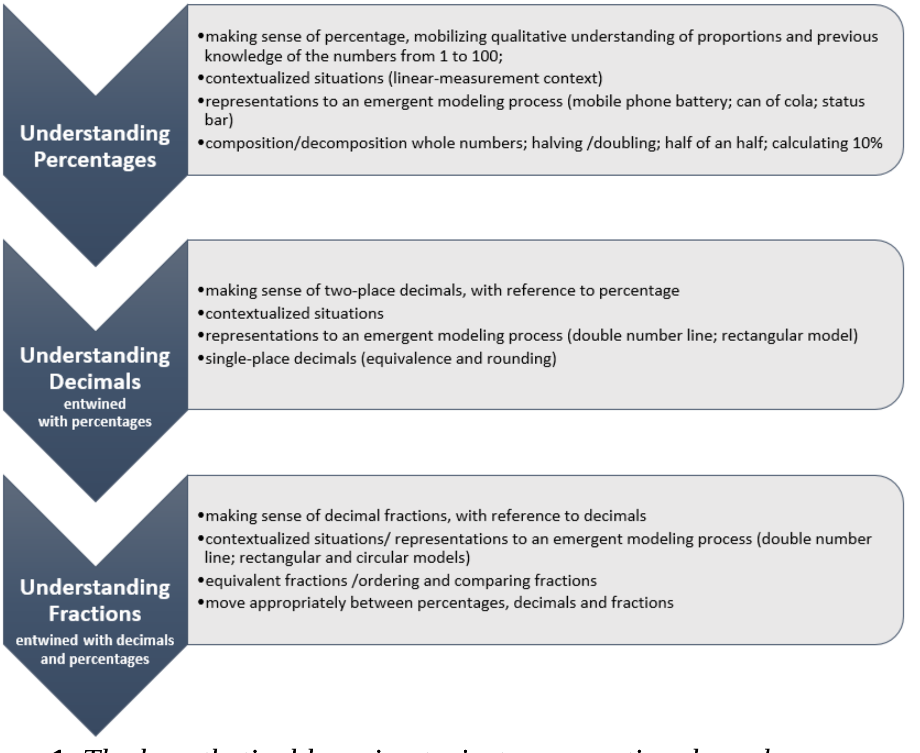

Figure 1. *The hypothetical learning trajectory on rational numbers.*

The first stage entails understanding of percentage. Then two-place decimals are introduced. Finally, fractions are the focus, concerning the use of different representations, in an interchangeable manner. The pedagogical dimension attempts to account for the means of supporting the co-participated learning process, as it occurs in the social context of the classroom.

Participants and Ethics

It took place spread across two 3-month periods, with the same classroom, in a public elementary school in Lisboa (Portugal), where the first author was also the teacher. The first period of the classroom experiment happened when the class was in third grade[2] and the second when they were in fourth grade. To avoid potential conflicts that might arise associated with the dual role, teacher and researcher, and assure students' and families' protection, an informed and voluntary consent was taken and students' individual and collective protection was ensured. Anonymity and confidentiality were guaranteed.

Data Collection

To establish reliability, and allow convergence, we choose to use multiple sources of data (Confrey & Lachance, 2000). The data corpus that we called for this paper consists of video recordings of whole-class discussions, students'

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[2] With 8 year old students.

written work, and teacher´s research journal. We analyzed students' interactions and written work when exploring tasks looking for evidences of students' mathematical learning development, as they participate in classroom community, by inferring the taken-as-shared intentions and meanings and interpreting the social use of representations.

Data Analysis

The analysis discussed in this paper focuses students' activity with some representations during the classroom experiment. This analysis has been carried out, according to a content analysis' procedure, which is still ongoing. We considered three interrelated categories as indicators of students' conceptual understandings of rational numbers as they handle percentage to: (1) encourage an emergent modeling process, (2) foster numerical relations and (3) support reasoning strategies. These analytical categories came from theoretical ideas crossed with elements that emerge from data, in a permanent interaction process, grounded in the details of the specific episodes.

Results

Preliminary Study

In this study, which took place at the beginning of third grade, students were presented to some contextualized tasks, which involved iconic representations related to percentage from daily life situations, as using smartphones, tablets or laptops. Our intention was to perceive how familiar students were with those representations and how significant they were establishing the relation between students' everyday life intuitive understandings of percentage and proportionality and coordinating it with students' strategies for manipulating whole numbers (Figure 2).

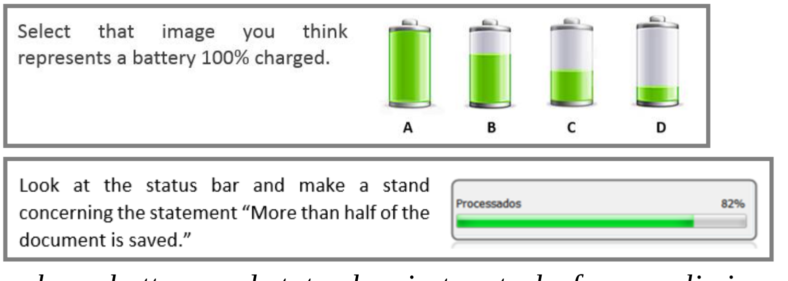

Figure 2. *Mobile phone battery and status bar in two tasks from preliminary study.*

Although percentage has not yet been covered in school, findings of this preliminary study support the idea that students are familiar with percentage terminology and appeared to have a good intuition about percentage from their everyday experiences. All students recognized the expression 100% and associate it, in the context of a mobile phone battery, to image A (Figure 2), the

battery, which is completely charged. The majority of students involved were also able to identify 50% as a half and justify using valid written explanations such as "... if it was 50% it would be a half, but as it is 82% you have more than a half saved" or "... because half is 50% and it is more than 50% saved".
In this sense, the classroom experiment was designed prioritizing the use of different representations associated to percentage from students' real-world situations. These representations were followed by others chosen to extend the understanding of whole numbers to the acquisition of knowledge about rational numbers, as an emergent and continuous modeling process of understanding numerical magnitude values' (Siegler et al., 2011).

The Emergent Modeling Process enhanced by Percentage

Understanding percentages. One of the representations used in the first stage of the classroom experiment was an iconic representation of a mobile phone battery (Figure 3) and students were asked to estimate the percentage represented.

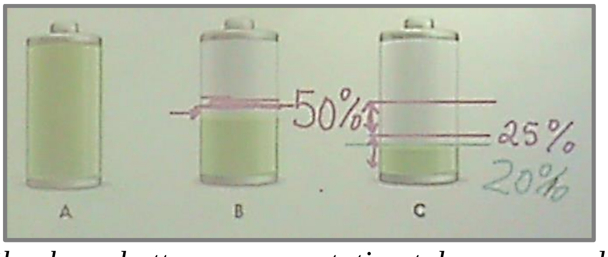

Figure 3. *Mobile phone battery representation taken as a model in a whole-class discussion.*

In small groups, some students came to the conclusion that the shaded part in battery C would represent 25% of fully charged and others said that it would be 20%. In the whole group discussion, students shared their arguments in order to justify their reasoning strategy, considering battery C. Dina's group calls for 25% and argues:

Dina: I saw 100 and made a half, which is 50 then it looked like a half again.

Dina used the successive halving strategy, a very common and useful multiplicative strategy, but inappropriate to this situation. Other students suggested a more precise measure:

Simão: That is not 25 percent.
Teacher: […] So, what do you think it is?
Students: 20 percent.
Teacher: Why? Marco.
Marco: Because it fits 5 times.

Students that stand for the 20% suggested that if the shaded part were iterated, it would fit the unit five times, so it should be 20% and not 25%, as it is shown in Ana's group written record, in Figure 4.

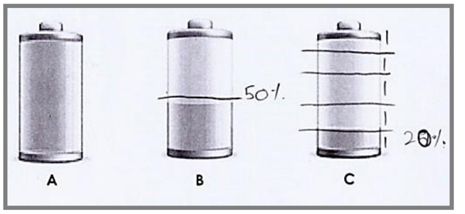

Figure 4. *Ana's group written record.*

The mobile phone battery is used and transformed, during a modeling process that supports students' reasoning strategies drawn on division and laid out on numerical relations. A measure interpretation allowed students to see the unit, represented by the full battery, as a length and the numerical value of percentage as a relative quantity of that length.

Other iconic representations were brought about from familiar contexts like the representation of a can of cola[3]. Students were invited to calculate, in small groups, the quantity of cola of the can when filled up to 50% and to 10% and to share their strategies with all class.

As long as the whole-class discussion took place, the iconic representation was used as a model for support thinking strategies as multiple symbolic representations emerged from students' strategies and helped them to organize their work, through interacting with one another (Figure 5).

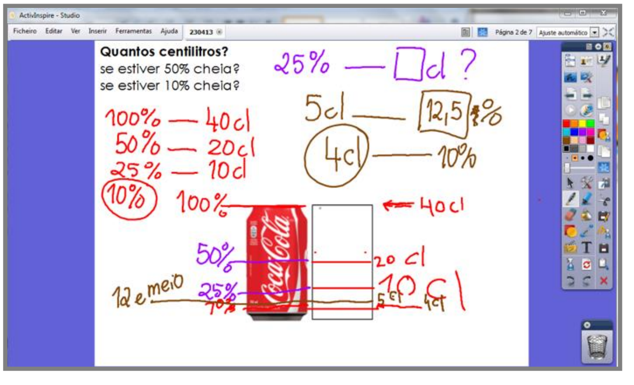

Figure 5. *A print screen from the smart board with the model being reconstructed during the whole-class discussion.*

The quantity of cola of the can, when filled up to 10% of its capacity, was non-consensual. Some groups pointed out five centiliters, others four centiliters. The first ones considered ten percent as a half of twenty five percent, arguing:

Horácio: We have been doing a half…and a half of 10 (centiliters) is 5 (centiliters).

The iconic representation becomes a model of the situation as students use it trying to get an efficient strategy. Figure 6 describes how a group of students

[3] The representation of a can of cola was tailored to consider its fullness (100%), rounded up to 40 centilitres, as the top of the can.

interpreted the relationships between quantities and the comparisons established, although their objective had been only partly achieved.

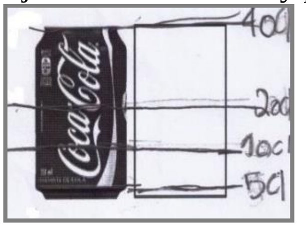

Figure 6. *Clara's group written record with a representation of a can of cola.*

Others could see that this halving strategy would be less efficient, highlighting that twelve and a half percent, and not ten percent, would be half of twenty-five percent and suggested another strategy.

Simão: As ten times ten equals to one hundred, four times ten equals to forty.
Teacher: What do you think? How did they think?
Heitor: I know what Simon did.
Ivo: I realized what Simon said.
Teacher: Okay. What about the rest of you? . . . So, looking at the scheme, Simon saw that one hundred percent are forty centiliters.
Heitor: Ten percent ten times equals one hundred percent.
Teacher: Okay.
Ivo: He is considering divided in ten times four.
Teacher: That's it. What is the number that you have to use ten times to equal forty?
Students: It's four.
Rui: Because four times ten equals forty.

Challenged by this new strategy, students completed one another's ideas. This intertwinement of ideas made the class realize that ten percent of forty centiliters were four instead of five centiliters. Students became aware of the structure of proportional relations, making comparisons between entities in multiplicative terms, supported by the ratio interpretation provided by percentage as it emerged from their engagement in the modeling process.

Status bar (Figure 7) was an additional iconic representation regarding percentage that has also been privileged. Students interpreted status bar in an easy way by analyzing its completion, a process that they were familiarized while downloading software or a file.

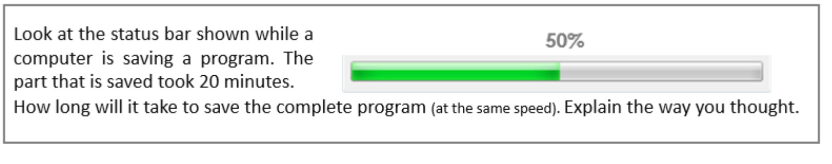

Figure 7. *A task that privileged status bar representation.*

Through this representation, students were able to establish numerical relations and to start seeing numbers as magnitude values'. In this task, the minutes taken to save the whole program were related to the amount of program saved shown in percentage, as Mafalda's group explains "The whole program took forty minutes to be saved because if a half is twenty, the double is forty". The explanation of this reasoning strategy seemed to make sense to the others as it expressed the multiplicative relation between half and double, keeping the ratio constant. Although not all students have clearly perceived it yet as a ratio, they saw that saving half of the program takes as much time as it takes to save the remaining half of the program, in the same proportion as 50% is to 100%. Thus status bar encouraged an intuitive ratio understanding, involving quantities of a standard unit, which can also be achieved when shifting to the double number line.

Understanding decimals. The double number line was crucial in this number system expanding idea with decimal's representation. In one of the tasks where double number line was used, students were invited to represent the value of the length measure of some classroom objects (Figure 8).

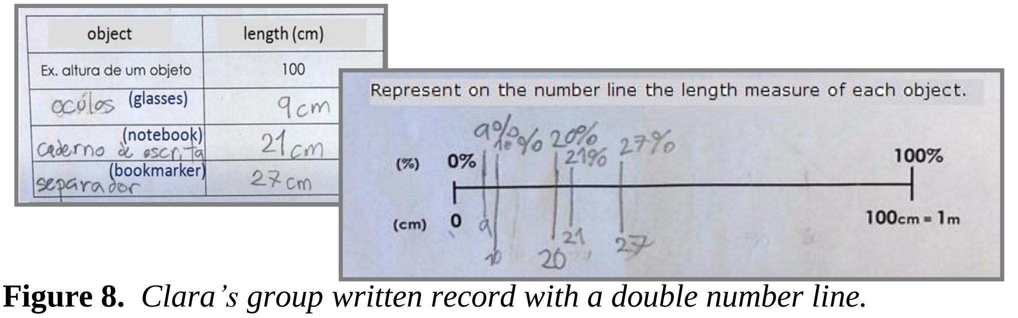

Figure 8. *Clara's group written record with a double number line.*

Students had to establish comparison relations between equivalent ratios, like percentage and the measurement of the length that represents a number in the number line, where identifying the unit was key. Number line seems to contribute to a growing understanding of the unit concept and its importance, as we can perceive in the passage of the dialogue amongst students during the whole-classroom discussion.

Bruna: The whole number line goes from zero to one hundred in percentage, but also in centimeters. When we think ninety-one centimeters we have ninety one percent.

Simão: Each one percent is a centimeter.

Teacher: In this specific situation because our unit is one hundred centimeters, that we can see as one hundred percent.

The double number line became a model as students use it to support their reasoning strategies. In another task, when changing the reference unit, decimal numbers emerged in the double number line (Figure 9) as students use it to explain the percentage of "fullness" of the unit, converting from one symbolic representation (percentage) to another (decimal representation) in a context-connected level of understanding.

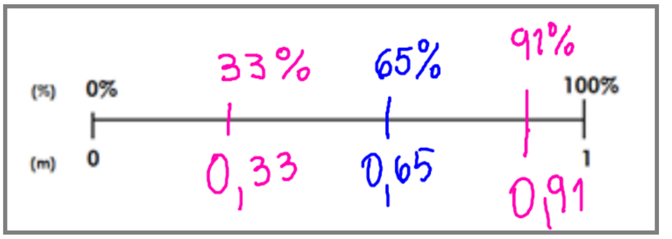

Figure 9. *Record from the interactive whiteboard done during a whole-class discussion.*

One of the challenges was to shift a measurement as 91 centimeters, corresponding to 91% of the unit considered one hundred centimeters, into a number line where the unit was then one meter.

Simão: It's not enough to make up one meter.

Teacher: How can we represent when we mean a part of the meter when we are not using the entire meter?

Clara: … You have zero coma ninety-one meters.

Teacher: Then, this is Clara's suggestion.

Clara: Because it's not enough to make up one meter, so it´s zero meters and then we write ninety-one centimeters.

Teacher: [Writes 0,91 at the number line].

In this situation, the decimal representation was convened to identify a quantity less than one, in a measurement context. Students consider one meter as the reference unit and centimeters are embodied as hundredths. Invoking percentage knowledge students establish relations between length and each decimal number magnitude value' considering its position in the number line. It indicates the percentage of the way between zero and one that each intermediate point lies in the number line, fostering a modeling process, as the double number line becomes a model for a more formal reasoning.

Understanding fractions. Later on, during the third stage of the classroom experiment, the double number line was also used in a context of a two hundred meters relay race with a four players' team, an activity in which students were effectively involved. In one of the tasks, students had to calculate the exact distance where each relay baton change took place and they could use the given double number line (Figure 10).

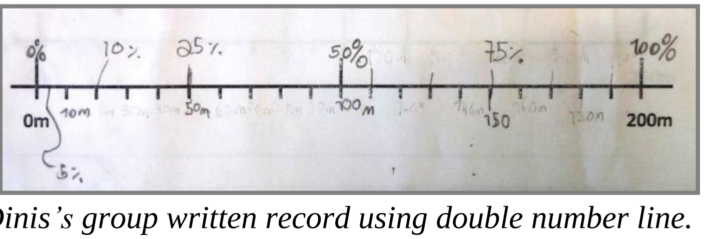

Figure 10. *Dinis's group written record using double number line.*

Most of the students could locate it correctly on the number line, partitioning the unit through a halving strategy, halving and then halving again, as Dinis's group written record (Figure 10) shows. They used percentage benchmarks to identify measurements as points in the number line which represented where each relay baton change, considering 100% as 200 meters.
In the same task, students had to determine if the given sentence "Each of the team players runs $\frac{1}{5}$ of the sprint relay" was true or false and justify using the number line. Most of the students agreed that the sentence was false and argued, during the whole-class discussion, saying:

Dinis:	There were only four people in the race.
Ivo:	Each one runs fifty meters.
Teacher:	And?
Ana:	If it were four people it must be from fifty to fifty and not from forty to forty. That would be five persons.
Teacher:	So, looking into the number line, is the claim true?
Heitor:	No, it´s one fourth. Each one runs twenty-five percent, not twenty percent.

According to students' argument, they can relate the number of players with the parts into which the unit must be partitioned, relying on the idea of fair sharing and associating with a percentage value. Some students, as Dina's group written record in Figure 11, represented each stage of the sprint relay considering both situations, if each player had to run one fourth or if each player had to run one fifth, in order to compare both, and trying to justify why the sentence is false.

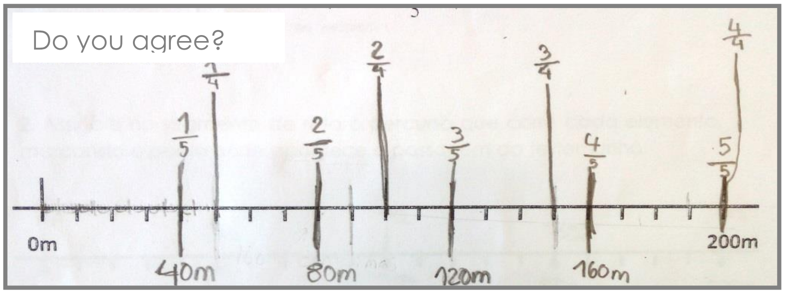

Figure 11. *Dina's group written record with the double number line*

Dina´s group, as other students, used the number line to locate fractions and compare the length they represent, reasoning about their sizes. Number line in this task was taken as model as it supported numerical relations and compare relative sizes of fractions, as they arise while making sense of a situation embedded in a reference context.

Discussion and Conclusion

With respect to the research question focused in this paper, we find evidence that students had actively constructed an emergent modeling process, supported by meaningful representations and empowered by percentage in contextualized situations.

The use of iconic representations associated to percentage, like a mobile phone battery or a status bar were shown to be useful when explored and transformed as models by students, emphasizing numerical magnitude values' understanding (Siegler et al., 2011). Also, the double number line as a representation of numerical magnitude when used by students became a linear-measurement model for looking at percentage, and decimal and fractional amounts, as measures of length (Moss & Case, 1999). In their arguments, students used those representations to make sense and solve problems, connecting such representations with one another. Students moved from natural language to the language of percentage, from the language of percentages to the language of decimals or to the language of fractions in a meaningful way, namely involving reasoning about the size of numbers represented. They see that 91% can be a decimal unit as 0,91 and that 25% can be reinterpreted as $\frac{1}{4}$ pursuant to a different context, taking the first steps into the interchangeability of representations (Moss & Case, 1999). An emergent modeling process, supported in multiple representations, as meant by Gravemeijer (1997) seems to have helped students to see how whole numbers, percentages, decimals, and fractions may relate in a structure of relations and familiar patterns, calling for an extension of number sense from whole numbers to rational numbers.

Data analysis highlight the role of percentage for learning rational numbers as it enabled an approach to multiplicative situations**.** Describing comparisons in multiple ways, students explored relationships, enhancing an introductory step to a conceptual understanding of rational numbers (Lamon, 2007; Parker & Leinhardt, 1995). Students began to have a grasp of rational numbers' sub constructs less common in this stages of elementary years, as ratio and measure, as Parker and Leinhardt (1995) suggest. These emphasized the concept of unit, allowing students to determine numerical relationships of order, comparison or equivalence. Also provided some evidence of a growing flexibility of students' reasoning strategies, as composition/decomposition, halving/doubling or unitization/reunitization.

Further research is required to gain a more complete understanding of how percentage, supported by multiple representations, can be an anchor to foster students' conceptual understandings of rational numbers. However, this study strengthens that working on percentage, through an emergent modeling process, students may use their experience with whole number computation and attempt to make sense of numbers and their relationships, necessarily embedded within the rational number system as a whole, entwined with learning of decimals and fractions. This seems to contribute to the emergence of a co-participated learning (Pontecorvo et al., 2005) on efficient reasoning strategies and a developing awareness of the rational numbers' relationships, among the classroom community.

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Authors:

Helena Gil Guerreiro
Agrupamento de Escolas Braamcamp Freire, UIDEF, Instituto de Educação, Universidade de Lisboa
E-mail: hg@campus.ul.pt

Lurdes Serrazina
Escola Superior de Educação de Lisboa, UIDEF, Instituto de Educação, Universidade de Lisboa
E-mail: lurdess@eselx.ipl.pt