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How to Prevent from Regarding Mathematics as Algorithm: A Study on the Beliefs of Mathematics Learning by Clinical Interview

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Twenty-nine 7th grade students were questioned about the rationales of algorithms. The results showed that most students failed to interpret algorithms with the rationales; the simpler the rationale of an algorithm is, the more easily it is ignored.

Key words: conceptions of mathematics, algorithm, rationale for algorithm.

Introduction

The concepts of mathematics derive from our previous experience in mathematics activities and influence the ways in which we work in mathematics (Andrews & Hatch, 1999). The acquisition of the proper concepts of mathematics would improve the mathematics behavior and itself is also an important aim that mathematics education must achieve; that is, the acquisition of the proper concepts of mathematics is not only the method, but also the important aim (Huang, 2002).

It has widely been realized that the concepts of mathematics are important, but some fundamental problems about the research on it still need to be discussed. In the related literature, almost all the views referred to mathematics were involved in the term ‘the Conceptions of Mathematics.’ Is it proper? Even so, should we differentiate between which is primary or more essential and which is secondary or has less influence on mathematics activity? These problems are concerning the limiting of the research topics. On the other hand, how can we probe individual concepts of mathematics inside? Is a direct inquiry effective? Which method is more effective?——these problems are concerning the selection of the means for research.

The Epistemological Beliefs about Mathematics

Many views were listed under the term "the Conception of Mathematics", such as: the mathematics epistemological beliefs, the views of mathematics knowledge, the views of mathematics teaching, the views of mathematics learning, the views of social situation about mathematics, the self-concepts concerning mathematics and so on (Liu & Chen, 2002; Liu & Zhang, 2003; Xu, 2006). There were more subordinate concepts, for example, the views of mathematics concerning symbols, the views of mathematics concerning operation, the views of mathematics concerning thinking, the views of the practicability of mathematics. Surely, all these views were related to mathematics, nevertheless, only in this sense, they were listed under the term "the Conception of Mathematics". The differences among them are greater than their similarities, for example, comparing the mathematics epistemological beliefs with the self-concepts concerning mathematics, the epistemological beliefs are the views of the psychological generating of the mathematics knowledge and about how to get them, and influence the way in which to learn; the self-concepts are about whether students were competent in learning mathematics, that is the 'Self-Confidence' and 'Self-Efficacy,' and influence confidences to learn mathematics. In short, the former relate to the ways while the latter relate to the confidences. Two categories are significantly different, but usually simply placed together just because they are all referred to with the word 'Mathematics.' We should divide them in order to advance research and access to the essential of the conception of mathematics.

It is common that they were simply placed together in the related empirical studies (Liu & Chen, 2002; Liu & Zhang, 2003; Xu, 2006). 'What is mathematics?' 'What is the use of mathematics?' 'Why do you like or not like mathematics?' 'What is the mathematics teacher in your mind?' 'How do you learn mathematics better?' These questions paid attention to all sides and always appeared in the same study. The data from it, of course, were all-sided, simple and superficial.

Now, how to avoid this? How to differentiate between primary and secondary? It is a problem about how to limit the research topics and also about how to the anchor the point of research. It should be not difficult, if we remembered the point of perspective from which we research the conception of mathematics.

The concepts of mathematics were discussed in the background of mathematics education, and the aim of it was to advance students' mathematics learning. The learning is primarily a process of cognition. The characteristic of the essential rooted in the nature of mathematics curriculum, and many mathematics educationalists' discussions were initiated from this fundamental problem.

As the end products of mathematics activities are being formalized, mathematics knowledge mainly embodied a set of algorithms; but the whole of mathematics is not merely a set of algorithms. Freudenthal (1973) pointed out: 'Algorithms are not in themselves automatisms, though by means of routines they make automatisms possible. This is their rationale, but is also the reason why they may endanger instruction.' Algorithms should be mastered and automated, but it must be avoided that mathematics be simply regarded as a set of algorithms (Such latent danger is called 'Latent Algorithm Equaling Danger' for short).

This is a dilemma. The whole history of mathematics is just the process of problem-solving and discovery. During this process, methods were found and formalized and became simple algorithms. But the algorithms are not artificial, and have their rational basis (i.e. the rationale for algorithm), therefore, besides algorithms, the whole of mathematics also involves other two elements: the process of discovery of algorithms and the rationale for algorithms, in a word, every algorithm has its discovery process and the rationale for itself.

However, the algorithms are relatively independent. Firstly, algorithms and their rationales could be detached from each other; that is, we may only be concerned math algorithms, for example, old say we need not learn the nature of the equation, and imitate 'Transposition' directly. Secondly, the algorithms could be separated from their discovery process. When an algorithm which was discovered is used again to solve problem, it's not necessary to replay its discovery process.

The relative independence of algorithms make it probable that people hand over the algorithms directly and ignore or shorten the discovery processes. This is the reason why the 'Latent Algorithm Equaling Danger' exists.

Is the mathematics learning a process of discovery or handing over? This problem is an essential problem in whole mathematics education, and this essential problem also concern the epistemological beliefs about mathematics, that is to say, it is the essential problem on which the studies of the concepts of mathematics should focus.

From the discussion above, it is obvious that the participant of the concepts of mathematics is not isolated. It is closely related to the fundamental problem in the whole of mathematics education. When we conjoin them, it's easy to grasp the essential in this field: the epistemological beliefs of mathematics are the core in the studies of the conception of mathematics.

When remembering the point of perspective from which we research the conceptions of mathematics, and making it related to the fundamental problem in whole mathematics education, we could ultimately avoid the idea that all ideas concerning mathematics were involved in the term "Conception of Mathematics".

The Clinical Interview

In most related empirical studies, there are usually three ways to probe the conceptions of mathematics.

Ask Directly

This includes open-ended questions (for example: What is mathematics?) and the options- limited questions (for example: Mathematics is accurate ---- right, wrong, I do not know). Regardless of the way, both the inquiry modes are directly similar. There is a basic assumption that the individual could be aware of the concepts of mathematics clearly and express it by speech directly. However, this is necessarily. The concepts of mathematics (especially Epistemological Beliefs) usually are implicit (Andrews & Hatch, 1999), and embodied in the mathematics behavior, and difficult to be aware of by oneself clearly. Of course, the implicit conceptions can be accessed through analysis, or inference. Some researchers have analyzed the materials from the inquiry directly, but most of them are described superficially. For example, to the question 'What is mathematics?' There is an answer: Mathematics concerns the number, calculation and algorithms. An answer like this, in fact, cannot represent anything.

Inquiry with Hypothetical Situation

This way was created by Koub and McDonald (1991), and many researchers (Huang, Lin, Huang, Han, Wang, 2003; Xu, 2006) have used this method in their studies (for example: One day you bought the newspaper, and then estimated the number of words of the front page of the newspaper. Do you think that means you are doing math? This way only pays attention to the extension of mathematics that is about the problem: what is mathematics, what is not mathematics. The materials which are obtained by “Inquiring with hypothetical situation” were mainly used to describe the participants’ views about the extension of mathematics as to whether it was narrow or broad.

Clinical Interview

Piaget made this method perfect. This method includes the following elements: first, activities: experimenters let the participants complete a certain task; second, inquiry: during or after the activities to complete the task, experimenters ask the participants crucial questions; third, inferring: experimenters speculate the psychological process from activities and inquiry, and then from this explore the inner mind structure and character. Clinical interview could access inner psychology. In this sense, it is better that any other method.

In recent years, educationalists have attempted to change the traditional way of evaluation of mathematics that only focuses on the results of mathematics activities, and emphasize the multi-evaluation, including the method of interview that can delve deeply into the inner mind process (not just the results). Some scholars used the Clinical Interview way to assess the mathematical achievement of students in their studies, and also explored the concepts of mathematics of students. In this research, they interviewed students with a boat renting problem, and found that some students attempted to look for a ready-made formula that could be applied directly, whose conceptions was ignescent (Wong, Lam, Wong, Law & Sun, 2006).

The concepts of mathematics are implicit, they even are thought to only be accessed via inference by some scholars (Andrews & Hatch, 1999). The implicit conception is always embodied in the mathematics activities and the processes of thought of mathematics problems. The clinical interview concerns the activities and the inner processes. In this sense, we say that the clinical interview is the preferred method in the exploration of the conceptions.

Method

Based on the discussion above, it is obvious that "how to avoid 'Latent Algorithm Equaling Danger'" is an inherent and fundamental problem in mathematics education. It is rooted in the nature of the mathematics curriculum. Many mathematics educators have put forward solutions; interestingly, their solutions are focused on problem-solving and discovery. In order to prevent students from seeing mathematics as set ready-made and external algorithms, mathematics educators have advocated replaying the processes of discovery of mathematics in the mathematics classroom. Polya (1965) said: "For efficient learning, an exploratory phase should precede the phase of verbalization and concept formation……Let the students discover by themselves as much as feasible under the given circumstances." Freudenthal (1973) said more directly: 'The best way to understand the trick is to discover it.'

For more than half a century, mathematics educationists have emphasized that through the use of problem-solving methodology to discover algorithms again and again, mathematics classrooms have improved. The ways in which the algorithms were introduced have been improved to some extent, but the method of illustrating algorithms still exists in classroom, especially in China. The method of illustrating algorithms (i.e. selecting a proper example, and then interpreting the algorithm), in fact, shorten the discovery process, so that "students could go through smoothly the paved path" (Zhang, Li & Li, 2003).

In such a classroom, what concepts of mathematics would the students acquire? Specifically, would students regard mathematics learning as receiving a set of outside algorithms, or seeking for understanding of algorithms?

According to the analysis above, this research will focus on the "Latent Algorithm Equating Danger," specifically, to study the following problem: would students receive or understand algorithms? Clinical interview will be adopted. Students will use inquiry with the algorithms which have been learned (for example: Transposition), in order to make sure whether students could interpret with rationale these algorithms (for example: the rationale for the algorithm Transposition is that subtracting the same item from the two sides of equation at the same time; one side offsets and another side left opposite one in symbol of '+' or'−'). If students were sensitive to the rationale for algorithm, they could provide a proper explanation, and it should show that the students seek the internal understanding actively. Conversely, if they did not provide a proper explanation, it should show that students regard mathematics learning as receiving a set of external algorithms.

Participant

A seventh grade class was selected, with 29 students, 13 boys and 16 girls. Every participant was interviewed by the same experimenter in a single day through one-to-one interviews. The seventh grade students are at the conversion phase of the school stage, and if they have previously learned certain concepts of mathematics learning, then the conceptions would impact the mathematics learning in middle school stage.

Materials

The experimental materials were selected from the mathematics book that the participants have studied (Transposition, Canceling Denominator). The experimenter asked the participants the questions about the reason for legitimacy of the formalized algorithms (i.e. rationale for algorithm).

Material 1 about "Transposition"

$$\underset{\text{(Equation I)}}{x+x+30+0.5x+15=180} \implies \underset{\text{(Equation II)}}{x+x+0.5x=180-30-15}$$

The algorithm and the rationale for algorithm in material 1. Algorithm: Any item is moved from one side of an equation to another side, and its symbol of '+'or '—' is changed.

The rationale for an algorithm: when adding or subtracting the same item from the two sides of equation at the same time, the equation is invariant. Specifically, after moving from left to right of the equation, '+30' is changed into '−30'. The reason of the change is that both sides of equation were subtracted "30"; at the left, the '+30' was offset by '−30' and the right of the equation was subtracted by '30', so '−30' appeared.

The inquiry about material 1. The positive inquiry: what have changed from Equation I to Equation II ? After moving from left to right of the equation, '+30' was changed into'−30', and why?

The negative inquiry: would be it feasible if we didn't change the symbol of '+'or '-'? What would happen if we didn't Change? And why?

Material 2 about "Canceling Denominator"

$$2x\ 5\% + 5x\ {\cdot}75\% + x=180\ 80\% \Longrightarrow 2x\ 5 + 5x\ {\cdot}75 + x= 80\ 80$$

(Equation I) (Equation II)

The algorithms and the rationales for algorithms in material 2. There are two levels of algorithms and rationale for algorithms----

Level 1: The algorithm: 'Canceling Denominator '

The rationale for algorithm: The reason of 'Canceling Denominator' is that if two sides of the equation are multiplied by the same number (i.e. Denominator), the equation is invariant.

Level 2: The algorithm: "Multiply Each Item by the Same Number"

The rationale for algorithm: the reason of "Multiply Each Item by the Same Number" concerns the distributive law of multiplication

The inquiry about material 2. Positive inquiry: Is it right to change Equation I into Equation II? Why must the third item at the left also be multiplied by 100? Negative inquiry: Would be it feasible that the third item wasn't multiplied by 100? What would happen if the third didn't multiplied by 100? And why?

Procedure

There were no learning effects for progressive relationship among the interview materials. Interview conducted from Material I to Material II, and presented materials first, and then inquiry. The whole interview process was recorded, and after the interview, the record was transformed into text. After the interview, the experimenter returned to the classroom and taught all students the rationales for algorithms.

Results and Analysis

Table 1

The General State and the Test the Significance of Difference between Participants Who Explained with the Rationale and Who Might Understand the Rationale

	General State		Difference Test			
	Explained with the Rationale		Might Understand the Rationale			
	N	Proportion	N	Proportion	Z	P
Material 1 (Transposition)	6	6/29	29	29/29	6.18	<0.001
Material2 (Canceling Denominator)	14	14/29	29	29/29	4.50	<0.001

During the interview, the experimenter asked the students questions about the algorithms that they had learned before (Transposition, Canceling Denominator), on both positive and negative aspects (for example, questions about transposition: Why should'+30' become '−30' when it was transposed from the left to the right of the equation? What would happen if we didn't change the symbol of '+'or'−'?), in order to make sure whether students could explain the legitimacy of the algorithms with the rationales. The results showed that on Material 1 (Transposition), only 6 students could make an appropriate explanation with the rationale, while the other students were not sensitive to the rationale as if the algorithm "“Transposition” were only an external and man-made rule for them. On Material 2 (Canceling Denominator), there were 14 students who make appropriate explanations with the rationales.

After the interview, the experiment or interpreted all the student rationales for “Transposition” and “Canceling Denominator,” so that all the 29 students (100%) might understand them.

Here is a difference: with the teaching of the rationale all the students might understand them, but part of the students, less than half, could make appropriate explanations with a rationale during the interview, that is, between understanding with experimenter’s interpretation (100%) and interpreting by the students themselves during interviews (less than half), no matter whether Material is I or II, the differences were significant (p<0.001). This showed the possibility of understanding of the rationale does not relate to the sensitivity of the rationale, that is, if students understand the rationale when the teacher teaches it orally, that is one. This “understanding” is with a strong meaning of ‘feeding’, while students’ seeking the legitimacy of the algorithms is another.

Error Types of the Participants in Interview

Table 2
Error Types of the Participants in Interview

	Could not Explain	Explained by ‘Objective’	Explained the Algorithms with Algorithms themselves
Materials 1	11	5	7
Materials 2	10	3	2

Errors made by the participants in interview can be sorted into the following 3 Categories.

Could Not Explain

Facing Material 1, all the students were clear about the algorithm: changing the symbol of “+” or “−”when transposing. But 11 students could not give any explanation when they were asked the questions ‘why should we change the symbol of '“+”or“−”?’and ‘what's the reason of changing the symbol of “+”or“−”?’

The same phenomena appeared in interview about the ‘Canceling Denominator’. Canceling ‘%’was by multiplied by‘100’, but there was an error, that is, the third item ‘X’ at left should be multiplied by‘100’when Equation I was transformed into Equation II in Material 2. All the 29 students could find the error, but 10 students could not explain when they were asked ‘What's the reason?’

Explained the Algorithm by Man-made Objects

Students explained the algorithms with man-made objects, and some reasons cited were ‘In order to calculate easily,’ ‘In order to get the answer.’ When students were questioned on the negative aspect (What would happen if we didn't change the symbol of ‘+’ or −’ while transposing? What would happen if not every item was multiplied by the same number while removing the denominator?), some students also answered on the negative aspect: ‘It would be wrong,' and could not explain with the rationale. When asked 'Why would it be wrong?" Some students were unable to give any answer.

Case 1 The Response to the Question of 'Transposition' by Participant A
Showing Material 1 (see above 2.2 materials)
Experimenter: What is changed when'+30'was moved from the left to the right side of equal sign?
Participant A: The symbol of '+' was changed.
Experimenter : Why would it changed?
Participant A: In order to calculate easily. The unknowns are transposed to left side, and constants are transposed to right side.
Experimenter : Then, why should we change the symbol of'+'?
Participant A: We must change it while moving.
Experimenter : Why? What is reason?
Participant A : I don't know.
Experimenter: What would happen if we didn't' change this symbol of'+'?
Participant A: Then we would not get the right answer. It would be wrong when that result was put in.

Here, all the participants were able to tell the man-made purpose when he (or she) was asked by the experimenter for the positive aspect or the negative aspect.

Explained the algorithms with algorithms themselves

When students were asked the problem: 'Why should "+30" become "-30" when it was transposed from the left to the right of the equal sign?' there were 7 students who explained the algorithm with an algorithm itself in a vicious circle.

Case 2 The Response to the Question of 'Transposition' by Participant B
Showing Material 1 (See above 2.2 Material)
Experimenter: '+30' has been transposed from the left side of equal sign to the right, what was changed?
Participant B: Symbol of '+' was changed.
Experimenter: Why should we change it ?
Participant B: It would be changed if you transposed.
Experimenter: Why? What is the reason?

Participant B: It would be changed if you transposed.
Experimenter: What is the reason to change the symbol of '+' or '−'?
Might we not change it?
Participant B: (could not answer)

Here, the participant always explained the algorithm with an algorithm itself when he was questioned once and again, positive or negative, and failed to realize the rationale for legitimacy of algorithm (i.e. minus the '30' on both sides of equation). The same phenomena also appeared in the interview on 'Canceling Denominator.'

The differences between the Responses of the Participants on the Material 1 and Material 2

Table 3
The Number of the Participants who Explained with the Rationale or not, and the Test of the Significance of Difference

Material 1		Material 2		Difference Test	
Explained with Rationale		Could not Explained with Rationale		X^2	P
6	23	14	15	4.88	0.05

As mentioned above, in Material 1, the rationale for algorithm of Transposition is that adding or subtracting the same item from the two sides of equation at the same time, the equation is invariant. In Material 2, there are two levels of rationales. First, the rationale for the algorithm of Canceling Denominator is that the two sides of the equation multiplying by the same number (i.e. denominator), the equation is invariant; second, and the rationale for algorithm of 'Each Item Multiply by the Same Number' is concerning the distributive law of multiplication. Comparing the two rationales, the rationale in material 1 is easier than in material 2. The results of the interview showed: the simpler the rationale for algorithm is, the more easily it is ignored; in other words, the algorithm in which the rationale is simpler, it is more probably regarded as a set of external rules to receive.

Discussion

The State of the Conceptions about Mathematics Learning of Seventh-year Students

As mentioned above, no matter whether the inquiries were on the positive aspect (‘why should‘+30’become‘-30’when it was transposed from the left to the right of the equal sign? What is the reason?) or negative (Might we not change it? What would happen if we did not change it? Why?); the experimenter always asked about the rationales which is the reason for the legitimacy of algorithms.

During the interview, most of the students could not back up the rationales from the algorithms. They knew the reasons for legitimacy of algorithms with the teacher's explanation, that is, they had been familiar with the algorithms and their reasons for legitimacy (i.e. rationales), and then placed the rationales aside.

After the interview, the experimenter returned to the classroom to explain the rationales so all the students could understand them. However, during the interview, the reason for the legitimacy of algorithms was completely outside the scope of attention on the part of the students. Moreover, the simpler the algorithms are, the more easily they are regarded as a set of ready-made rules to receive. This showed that for these students, mathematics learning is considered as receiving the algorithms as set rules outside themselves.

The Main Causes that Lead to the Concepts of Mathematics Learning of Reception Passively

For the acquirement of mathematics knowledge, understanding means discovery and construction, and it is an inner process for an individual. It cannot be obtained directly by handing it over. In current primary and secondary school mathematics teaching materials in China, there is mostly one way to illustrate algorithms, in fact, the method is to hand over the learning directly. Under the influence of these teaching methods, students could not generate a real inner conflict. Therefore, It would be impossible to explore actively and seek genuine understanding. Students receive a set of rules outside themselves passively.

To change this state, many things are necessary: to improve mathematics classroom, to change the way of teaching algorithms by illustrating, to make the students experience the discovery process more completely in order to cultivate the habit with which they would explore actively. In our research, we found most of the students regarded mathematics learning as a process of receiving passively; the more simple the algorithms are, the more easily they are regarded as a set of ready-made rules to receive. In order to avoid the above state, it is necessary to change the way of ‘Illustrating Algorithms’ in the current primary and secondary school mathematics classroom.

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