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Chinese Teachers' Knowledge of Teaching Multi-digit Division

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The study investigated Chinese teachers' content and pedagogical knowledge and measures of multi-digit division. 385 elementary school teachers from the 1st to 6th grades at 37 schools in six cities/regions in four provinces in China participated in the study. The findings showed that Chinese teachers' knowledge of multi-digit division had multiple dimensions and they were able to use estimation as a main approach to foster students' conceptual understanding and computation fluency in learning multi-digit division. The study provided a rigorous and empirical way to measure teachers' pedagogical content knowledge and provided statistical evidence of how teachers' content knowledge connects to their pedagogical knowledge in the three areas of content and six areas of pedagogy in multi-digit division.

Key words: elementary mathematics, professional development, problem solving, in-service mathematics teacher education.

Introduction

Recently, math educators have been enthusing about conducting research involving East Asian countries-- since students of East Asian, such as Chinese students, show outstanding performance in various international assessments (Stevenson, Chen, & Lee, 1993; Stevenson, Lee, & Stigler, 1986; Stevenson & Stigler, 1992; Zhou & Peverly, 2004; Zhou, Peverly, Boehm, & Lin, 2000). Culture, school organization, number-word systems, the content and curriculum had been proposed as factors causing the achievement gap (e.g. Geary, Siegler, & Fan, 1993; Miller, Smith, Zhu, & Zhang, 1995; Sutter, 2000). Besides all the above factors, teacher knowledge was argued as another issue that shaped the difference. Recent increased studies indicate that the knowledge required to teach mathematics well is specialized knowledge (Hill, Rowan, & Ball, 2005), content knowledge (Ma, 1999), pedagogical knowledge (Darling-Hammond, 2000), or pedagogical content knowledge (PCK) (An, Kulm, & Wu, 2004; Shulman, 1987). Thus far, most research studies have reported mathematics teachers' knowledge in general, and little research has focused on teachers' knowledge in a specific content with a wide ranging investigation. In addition, current research is struggling to find a measurable way to evaluate mathematics teachers' knowledge, especially to measure mathematics teachers' pedagogical content knowledge empirically. This study sought to investigate Chinese elementary school teachers' knowledge of how to teach whole number multidigit division and to explore empirical tests to measure teachers' pedagogical content knowledge in multidigit division.

Theoretical Framework

Teachers' Knowledge of Whole Number Division

A Recent National Mathematics Advisory Panel **(**NMAP**)** Report (2008) has recommended the critical foundations for algebra. One of such critical foundations is to achieve proficiency with whole numbers. To help students succeed in learning whole numbers, the NMAP indicates that teachers' mathematical knowledge is important and "teachers must know in detail the mathematical content they are responsible for teaching…" (p.37). During the last two decades, several studies focused on teachers' knowledge of whole number division and they found teachers had insufficient knowledge on division in their studies: (a) teachers failed to connect their understanding of division and the relationship between word problems and division computation procedures (Simon, 1990, 1993); (b) teachers got correct answers on division questions, and only a few could provide conceptual explanations for the mathematical principles and meanings of division (Ball,1990); (c) teachers' partitive dispositions toward division exacerbated many difficulties that quotative dispositions towards whole number division with remainder may resolve (Zazkis & Campbell,1994). Carpenter and colleagues (Carpenter, Fennema, Peterson, & Carey, 1988; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Franke, Carpenter, Fennema, Ansell, & Behrend, 1998) found significant relationship between teachers' knowledge of student thinking and students' achievement in the domain of whole-number arithmetic. Thus, teachers' difficulty of division understanding might also be an explanation for the students' poor understanding of division: According to the National Research Council [NRC] (2001), two common U.S. division algorithms can create difficulties for students. One such division algorithm is that "the algorithms require students to determine exactly the maximum copies of the divisor that can be taken from successive parts of the dividend" (p.210). This determination requires good number sense on estimation and fluent computation for students. It also provides a challenge for teaching division with multiple digits. However, little research is available to shed light on effective ways of teaching division algorithms. NRC recommended two methods that are commonly seen in East Asian countries: Mental arithmetic and estimation. Mental arithmetic can lead to deeper insights into its learning (Beishuizen, 1993); although estimation is a complex skill, it can facilitate number sense and place-value understanding (NRC, 2001; Reys, Rybolt, Bestgen, & Wyatt, 1982).

Research on Teachers' Knowledge

Several researchers tried to define and develop the category of the teachers' knowledge as well as expertise for teaching mathematics. Shulman (1986) and Wilson, Shulman and Richert (1987) conducted series of studies and explored ideas about how knowledge influences teaching and they indicated that knowledge of how to teach content is as important as knowledge of content that affects teachers' effectiveness. They proposed three categories for mathematics teachers' knowledge: (a) content knowledge, which includes facts and concepts in a domain as well as why they are true and how knowledge is generated and structured; (b) pedagogical content knowledge, which includes representations of specific content ideas and the knowledge of what are difficult and simple points for students to learn a specific content; (c) curriculum knowledge, which includes the knowledge of how topics are structured within and across school years and the knowledge of curriculum material utilizing. Using Shulman's model as a base, Fennema and Franke (1992) discussed five components of teachers' knowledge: the knowledge of the content of mathematics, knowledge of pedagogy, knowledge of students' cognitions, context specific knowledge, and teachers' beliefs. Based on Shulman's work, An, Kulm, and Wu (2004) addressed pedagogical content knowledge as a connection between content and pedagogical knowledge. Ball and colleagues (Ball & Bass, 2000; Ball, Lubienski & Mewborn, 2001) further pointed out from their series of research that the ability of unpacking mathematics content into understandable pieces for students is another category of knowledge for teaching mathematics. Researchers also attempted to apply Shulman's theory on specific content topics: (a) equivalent fractions (Marks, 1992); (b) functions (Sanchez & Llinares, 2003); (c) decimal numeration (Stacey et al., 2003); and (d) algebra (Ferrini-Mundy, Senk & McCrory, 2005).

With many categories of knowledge posed by researchers, attention has been focusing on how to measure teachers' knowledge. Various qualitative methods were used to deeply explore how teachers use various strategies to explain and represent mathematical content to their students (An, Kulm, & Wu, 2004; Ball, Hill, & Bass, 2005; Ma, 1999). Quantitative methods were used by Hill, Rowan, and Ball (2005) to explore the relationship between students' mathematics achievement and their teachers' mathematics-related knowledge. Although many researchers used different instruments in their empirical approach to measure teachers' knowledge, few instruments could tap teachers' knowledge directly (Krauss et al., 2008). For example, how content and pedagogical knowledge relates to the pedagogical content knowledge remains noticeably absent from recent research. "And to date, scholars have not attempted to measure teachers' knowledge for teaching in a rigorous manner" (Hill, Schilling, & Ball, 2004, p.4).

Chinese Teachers' Knowledge

During last two decades, a number of different international studies indicated Chinese students' achievement and ability from K-12 on mathematics, except graphing, using tables, and open-process problem solving, all outperform U.S. students(e.g. Chen & Stevenson, 1995; Miura, Chungsoon, Chang, & Okamoto, 1988; Stevenson, Lee, Chen, Lummis et al., 1990; Stigler, Lee, & Steven, 1990). Teacher-related factors were argued as one of the important factors that caused the achieving gap between U.S. and Chinese students. Teachers' knowledge, along with curriculum and teaching organization can be assumed positively related to students' performance (Wang & Lin, 2005).

Various studies showed that compared with U.S. teachers, Chinese teachers had a better knowledge and understanding of the mathematics content they taught and had better knowledge on how to demonstrate the explanations flexibly with different methods to their students (Ma, 1999; Perry 2000; An, 2004). Several reasons were provided from different researchers to explain why Chinese teachers had more profound knowledge than U.S. teachers. Chinese teachers gained their knowledge from each other through school teacher group activities: Prepare their curriculum and plan lessons, observe and evaluate each other's lessons, and investigate students' learning together (An, 2004; Paine, 1997; Paine & Ma, 1993). Chinese teachers' knowledge was further increased through participation in effective professional development programs that were based on teacher's daily life and pertinent to the teacher's needs through school-based teaching research mechanism (Stewart, 2006).

Although research had studied Chinese mathematics teachers' content knowledge (e.g., Ma, 1999), and pedagogical content knowledge (An, Kulm, & Wu, 2004), most research studied teacher knowledge qualitatively through teacher case studies (e.g., Grossman, 1990), expert-novice comparisons (Leinhardt & Smith, 1985), and studies of new teachers (Ball, 1990; Borko et al., 1992); few were focused on measures of teachers' knowledge. The goals of the current study are to investigate Chinese teachers' knowledge of teaching multigit division, to explore a measurable way to gauge teachers' pedagogical content knowledge, and to examine the relationship between Chinese teachers' educational and teaching backgrounds and knowledge. The research questions asked were:

(1) What strategies of teaching multi-digit division are derived from Chinese teachers' content and pedagogical knowledge?
(2) How well is the teachers' content and pedagogical knowledge connected as pedagogical content knowledge?
(3) Is there any significant association between Chinese teachers' educational and teaching backgrounds and their knowledge?

Method

Participants

Participants were 385 elementary school teachers from the 1st to 6th grades at 37 schools in six cities/regions in four provinces in China in 2007. All teachers were assessed via a questionnaire on their content and pedagogical knowledge in mathematics instruction in a large-scale study. Table 1 shows the teachers' demographic information.

Table 1

Teachers' Demographic Information

Number of Teachers at Each Grade Level							Degree			Major		
Grade	1	2	3	4	5	6	HS	AA	BA	Mat	Oth	73

							Equiv.			h	er 27	
Freq (n)	56	66	53	86	69	46	41	221	122	109	188	46
%	14.5	17.1	13.8	22.3	17.9	11.9	10.6	57.4	31.7	28.3	48.8	13.4

Data Collection and Instrument

Data was collected via a questionnaire with nine parts on teachers' knowledge and professional development. The mathematics instruction content and pedagogical knowledge part included five problems in whole number division, algebra, measurement, geometry, and statistics. For whole number division with multiple digits, the teachers were asked to answer the question using their mathematical content and pedagogical knowledge: How do you teach 328÷41?

Data Analysis

Both quantitative and qualitative methods were employed to examine the research questions. The responses on strategies of solving and teaching the multidigit division from the teachers were categorized into three types and coded as CK – Content Knowledge (how to solve) in seven categories, PK - Pedagogical Content Knowledge (how to teach) in six categories, and PCK - Pedagogical Content Knowledge (Association between CK and PK) for comparing and analyzing data (Lincoln & Guba, 1985). Table 2 shows the categories and codes of these three types of knowledge in teaching multidigit division.

Each item in Table 2 was assigned a numerical number for statistical analysis. Chi-Square and Pearson Correlation tests were used to identify a statistical association between variables of teachers' content and pedagogy knowledge, and teachers' knowledge and their education and teaching background.

Since pedagogical content knowledge is defined in this study as a connection between content and pedagogical knowledge (An, Kulm, & Wu, 2004), the measures of pedagogical content knowledge focused on examining how and what type of content knowledge is associated with a certain type of pedagogical knowledge and the strength of such associations.

Table 2

Categories of Three Types of Knowledge in Multidigit Division

Pedagogical Content Knowledge (PCK)	
Content Knowledge (CK)	Pedagogical Knowledge (PK)
Estimate	Review and Scaffolding
Place Value	Students try & Teacher Explains
Algorism	Question
Esti- Place V	Situated Learning
Esti-Algo	Address Misconception
Place V-Algo	Other
All	

Results

Common Ways of Teaching Whole Number Division

Teachers' Content Knowledge

Table 3

Teachers' Content Knowledge of Multidigit Division

Methods	Frequency	Percent	Valid Percent	Cumulative Percent
Estimate	276	71.7	71.7	71.7
Place Value	22	5.7	5.7	77.4
Algorithms	47	12.2	12.2	89.6
Esti- Place V	6	1.6	1.6	91.2
Esti-Algo	25	6.5	6.5	97.7
Place V-Algo	6	1.6	1.6	99.2
All	3	.8	.8	100.0
Total (*n*)	385	100.0	100.0	

The results from Table 3 show that Chinese teachers have profound content knowledge with a variety of ways in solving 328÷41. About 90% of teachers mentioned one way of teaching it: 72 % of the teachers would use estimation to teach 328÷41, 12 % of the teachers would use algorism, and about 6 % of the teachers would use place value approach. However, about 10% of the teachers mentioned two ways of teaching whole number division. For example, about 7 % the teachers used estimation and algorism together for teaching this problem. Less than 2% of the teachers used a combination of estimation and place value or a combination of place value and algorithms.

Note. n = Number of responses

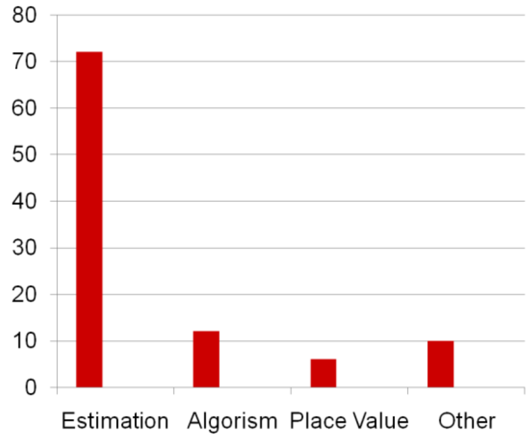

Figure 1. **Teachers' Content Knowledge of Multi-digit Division.**

Example of Teachers' Content Knowledge: Estimation and Place Value Methods

(1) Try out and Adjust quotient

Most teachers mentioned to estimate a quotient first, then try out the quotient and adjust the quotient. For example, Ms. Li, a 2nd grade teacher with 22 years of teaching experience and an associate degree said:

1. Consider 328 as 320, 41 as 40
2. First try out 320÷40
3. Adjust quotient

A fifth grade teacher, Mr. Zhang with 12 years of teaching experience and an associate degree in education echoed the try out a quotient and adjust the quotient strategy in estimation:

1. Try out quotient; consider 328 as 320 and 41 as 40.
2. 320 has 32 of 10s, 40 has 4 of 10s
3. From 32÷4 we can see the quotient

Ms. Shi has 26 years of teaching experience with only high school education agreed considering 41 as 40, but mentioned to use multiplication in estimation:

1. Consider 41 as 40 to try out
2. Look at what times 40 close to 320, 40×（ ）=320

3. Use 8 as try-out quotient

(2) Decide the Position of the Quotient, Try Out and Adjust the Quotient

A 6^{th} grade teacher, Mr. Zhou with 29 years of teaching experience and an associate degree in mathematics described his general steps of whole number division with multiple digits:

1. Decide the place of the quotient
2. Try out the quotient
3. Adjust the quotient
4. Write the process of finding the quotient

Another 6^{th} grade teacher, Mr. Ding, who has been teaching for 28 years with an associate degree in education, described his approach to finding the position of a quotient: have students analyze the problem, identify the divisor as a two-digit number, and then look at the first two digits of the dividend. Since 32 in the dividend is less than the divisor 41, meaning the division needs the third place, so the quotient will be written on the top of 8. Now think about that 4 times what numbers to get or close to 32, so the quotient is 8.

A 5^{th} grade teacher, Ms. Ma has teaching experience for 20 years and had an associate degree in education, agreed with Mr. Ding's idea of deciding the position of the quotient:

1. Have students view the divisor 41 as 40 to try out
2. Using two digits in the dividend 328 is not enough, so the quotient is in ones place
3. View 328 as 320 to get a quotient 8
4. 40 x 8 = 320. 320 is close to 328
5. 328 – 320 = 8. Therefore, $328 \div 41 = 8$

(3) Look at Digits of Divisor and Dividend

Ms. Ling, a fifth grade teacher with 22 years of teaching experience and an associate degree in education, used the place value method by looking at the numbers of digits in the divisor first:

1. Ask students: how many digits the divisor is?
2. Then ask: is it enough to use the first two digits to divide 41?
3. If not, use$328 \div 41$. What is the quotient?

Mr. Xu, a 4^{th} grade teacher with 29 years of experience and high-school education looked at the numbers of digits in the dividend first:

1. Use 40 to try quotient
2. Look at the first two digits of the dividend
3. If it is not enough, look at the first three digits of the dividend. Since the first two digits –32 is not enough, so the quotient should be at the ones place
4. Put the quotient at the place ending dividing

(4) Estimate Range of Quotient and Transfer Prior to New Knowledge

Pedagogical Knowledge

Single method. Table 4 shows that Chinese teachers were able to use different pedagogical approaches for teaching multidigit division: about 87% of the teachers would use review and scaffolding as a main focus of instructional strategies; 52% of the teachers would have students try quotient first and then the teachers explain it; 30 % of the teachers liked to use questioning strategy in teaching for multidigit division; about 7% of the responses indicated the situated learning strategy; close to 7% of the responses indicated addressing misconception during the teaching.

Table 4

Methods	Review and Scaffolding	Students Try & Teacher Explains	Question	Situated Learning	Address Misconceptions	Other	*N*
Freq (*n*)	333	201	117	25	26	10	38
%	87	52	30	7	7	3	5

Pedagogical Knowledge of Teaching Multidigit Division

Note. *n* = Number of responses

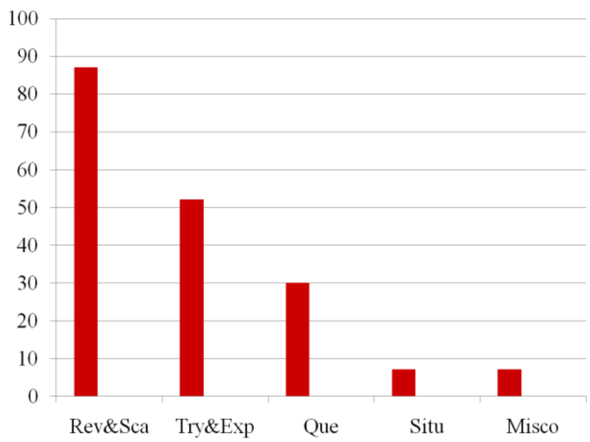

Figure 2. **Teachers' pedagogical knowledge of multi-digit division.**

Combined method. The data in Table 5 shows that 15.6 % of the teachers combined the review and scaffolding with the students try and then teacher explains method; 17.9% of the teachers combined the two ways with questioning strategy; less than 4% of the teachers liked to add addressing misconception with the first three methods or using other approaches.

Table 5

Combination of Teachers' Pedagogical Knowledge

Methods	Rev & Sca	St Tr &Th E	Re S_ St Tr	Re_St _Q	Re_St_ Q_M	Re_St_ Q_O	*n*
Freq (*n*)	151	31	60	69	13	11	385
%	39.2	8.1	15.6	17.9	3.4	2.9	*n*

Note. n = Number of responses

Example of Teachers' Pedagogical Knowledge: Single and Combined Methods

(a) Review and Scaffolding method – Transition of knowledge

A first grade teacher, Ms. Chu who has 17 years of teaching experience with an associate degree in Chinese mentioned her steps of review and scaffolding method:

1. To teach328÷41, I would have students estimate that 320 divided by 40 equals 8. It gives a range of the quotient.
2. Have students think about how to do 32÷4=8 in a vertical form and recall the algorithms
3. Last, give the new problem 328÷41. Ask students: Do you know how to do it?

Ms. Zhang, a 5th grade teacher with 32 years of teaching experience and a BA in mathematics explained how to help students transfer their prior knowledge to the new learning:

First, have students use a divisor of one digit (328÷4). If the number in the highest place of the dividend is less than the divisor, look for two places in the dividend. Whatever place is used in the dividend, put the quotient on the top of it. Each time, the remainder should be less than the divisor. Second, use the transition method to teach 328÷41. The divisor 42 is a two-digit number. If the first number in the two digits of the dividend is less than the divisor, how many places should we look at? Three places. Students will solve 328÷41 naturally using learned knowledge.

Mr. Rong, another 5th grade teacher with 20 years of teaching experience and an associate degree in Chinese summarized the transition method: Review division with one digit divisor; decide the position of a quotient; select the "try-out" quotient method, such as viewing 41 as 40, or 328 as 320.

(b) Student Try and Teacher Explain:

Ms. Tang, a 2nd grade teacher with 16 years of teaching experience and an associate degree in education said:

1. Students try out first
2. Students come up and show how to solve it
3. Teacher explains based on the student work

(c) Question

Ms. Wu, a 1st grade teacher with 12 years of teaching experience and an associate degree in education would like to use question strategies:

1. Have students self study 328÷41
2. Discuss their work with peers
3. Ask questions to students:
   a. What did you learn from interacting with peers?
   b. Why is the quotient in the ones place, not in the tens place?
   c. How did you try out quoticent?
   d. What is your try-out quotient?
   e. What is actual quotient?
   f. Why are the two not the same?

(d) Situated learning

Ms. Ye, a 4th grade teacher with 13 years of teaching experience and an associate degree in Chinese explained her view of teaching 328÷41 with situated learning: to teach 328÷41, create a real situation, have students use 328÷41 to solve the problem, and direct students how to use a vertical form to figure out the quotient of 328÷41.

(e) Address Misconceptions Based on Student Work

Ms. Fang, a 5th grade teacher with 12 years of teaching experience and an associate degree in education addressed misconceptions in teaching:

1. Have students self-study first
2. Point out students' errors
3. Have students self-correct errors
4. Summarize the correct methods of computation and emphasize the main parts needed to pay attention

Mr. Xing who only has one year of teaching experience at the 3rd grade, but has a BA in computer education agreed with Ms. Fang on addressing misconceptions: Use student prior knowledge to have students try to solve it. From students' work, find problems and errors. I will make corrections for errors one by one and teach students how to divide a three-digit number by a two-digit number.

(f) Review-Student Trying Out–Teacher Explains

Ms. Huang, a 6th grade teacher with 20 years of teaching experience and an associate degree in education, combined review, students try, and teacher explanation methods:

1. Review how to use three digits to divide by one digit method
2. Have students try out a quotient
3. Emphasis to divide from the highest place value of dividend. Use first two places in dividend; if it is less than divisor, use the first three places in the dividend.

(g) Compare and Select the Efficient Methods

Ms. Gao, a 5th grade teachers with 13 years of teaching and a BA in education would like to do the following ways:

1. Have students estimate based on their prior knowledge
2. Try to use the vertical form to calculate
3. Have students share their different ways of computation and explain their reasoning
4. Select the best method of solving

5. Ask all students to have a consistent written format
6. Stress the key points, exchange ideas, and generalize the methods
7. Cultivate students' self-checking habits.

Mr. Liang, a 4th grade teachers with 12 years of teaching experience

and an associate degree in education also would like to have students do 328÷41 based on their prior knowledge. He asked two students at different levels to come up to write their computation process on the blackboard. Then he would ask them: Why did you solve it in this way? What was your thinking? From the discussion, they find a good way and generalize it.

Other teachers, such as Ms. Liu, a 6th grade teacher with 12 years of teaching experience and an associate degree in Chinese would like to have students discuss the problem in a small group. Then have their group representative share their methods. The whole class will provide feedback and select the best method. The teachers will summarize and praise the students with the correct methods and encourage other students to learn these methods.

Pedagogical Content Knowledge of Teaching Whole Number Division

Table 6 shows that Chinese teachers were able to connect their content knowledge to pedagogy knowledge with multidigit division. Specifically, when using the estimation method for multidigit division, about 71% of the teachers would use review and scaffolding; 44% would have students try and then the teacher explain it; 23% would use questioning strategy; 3% would use situated learning; about 5% would address misconceptions. When using the place value approach to solve multidigit division, less than 9% would use review and scaffolding; and less than 7% would use other methods. When using the algorithms, about 17% would use review and scaffolding and 12 % would use having students try it first and then the teacher explains it; about 10% would use questioning strategies; about 4 % of the responses indicated addressing misconceptions as a main focus in instruction.

Table 6

Pedagogical Content Knowledge of Teaching Multidigit Division

Connection of Content and Pedagogical Knowledge	N	%
CK_Estim * PK_Review	272	70.6%

CK _Estim * PK_St_Teach	169	43.9%
CK _Estim * PK_Question	89	23.1%
CK _Estim * PK_SituLearn	12	3.1%
CK _Estim * PK_Adre_Mis	18	4.7%
CK _Estim * PK_Other	2	.5%
CK _PlaceV * PK_Review	35	9.1%
CK _PlaceV * PK_St_Teach	14	3.6%
CK _PlaceV * PK_Question	11	2.9%
CK _PlaceV * PK_SituLearn	2	.5%
CK _PlaceV * PK_Adre_Mis	2	.5%
CK _PlaceV * PCK_Other	3	.8%
CK _Algo * PK_Review	66	17.1%
CK _Algo * PK_St_Teach	46	11.9%
CK _Algo * PK_Question	40	10.4%
CK _Algo * PK_SituLearn	13	3.4%
CK _Algo * PK_Adre_Mis	15	3.9%
CK _Algo * PK_Other	7	1.8%

To investigate the relationship between content and pedagogical knowledge, Cramer's *V* was used. The analysis indicated a significant positive association between Chinese teachers' content and pedagogical knowledge (Cramer's $V = .33$, $p < .001$), which means that more content knowledge associated with strong pedagogical knowledge, resulted in high level of pedagogical content knowledge. This association can be reflected in a large effect size, *Eta* = .404.

Examples of Pedagogical Content Knowledge

(a) Pedagogical Content Knowledge in Estimation and Place Value

Ms. Sun, a 1st grade teacher with 13 years of teaching experience and an associate degree in Chinese addressed her view of teaching 328 ÷41:

1. Review 328 ÷40
   a. Student practice and make corrections

b. Ask students question: Why is the quotient not in the hundreds place when328÷4?

c. Now the teacher asks you to do328÷41. Do you have the confidence?

2. New lesson
   a. Have the whole class do 328÷41
   b. Provide feedback and explain
      i. Try out a quotient, use 40 to try it
      ii. Calculate, remember the rule of multiplication: 40×8 = 320
      iii. Make corrections
3. Practice for Reinforcement.

Ms. Sun's colleague, Ms. Li, a 3rd grade teacher with 17 years of teaching experience and an associate degree in elementary education agreed with Ms. Sun's methods, but s slightly different way:

a. Have the whole class do 328÷41

b. Explain the algorithms of computation
   i. Try out a quotient
   ii. Estimate the quotient, 41 is close to 40, use the rule of multiplication: 40×8 = 320, therefore, 8 is the quotient
   iii. Discuss the method

Mr. Yang, a 3rd grade teacher of 13 years of teaching experience and a BA in applied mathematics addressed his ways of teaching 328÷41:

1. Teach 328÷4 and 328÷40 based on student previous learning on one-digit divisor and two-digit divisor
2. First, create a situation, give the problem, and estimate
3. After students share their estimation, have them try to use a vertical way to solve the problem
4. Have students share methods and principle of computation.
5. Based on the above steps, have students use a vertical way to solve 328÷41.
6. In the process of solving, reinforce computation and estimation

Mr. Yu, a 4th grade teacher with 26 years of teaching experience and an associate degree in physics and chemistry major provided a teacher and student dialog about 328÷41:

Teacher: 41 is close to what number?

Student: 40.

Teacher: If we view 41 as 40, what should be a quotient?

Students will work independently and will share their solution upon finishing.

Teacher: What are the benefits to viewing 41 as 40?
Student: Quickly try out a quotient
Teacher: When a divisor is not a two-digit number with multiple of ten, how do you try out the quotient?
Students will discuss and summarize the methods and try out rounding.

Ms. Pei, a 3rd grade teacher with 13 years of teaching experience and a BA degree in PE responded with the following strategies in teaching 328÷41:

1. Have students practice 328÷4 and 328÷40
2. Ask questions: what is the highest place of quotient? Why?
3. Give students 328/41 to try
4. Based on students' practice, discuss methods of computation together, and then have focused practice.

(b) Pedagogical Content Knowledge in Algorism

A 6th grade teacher, Mr. Mao with 19 years of teaching experience and an associate degree in Chinese provided the following methods:

1. Review and scaffolding: oral practice: 120÷60? 240÷40=? 320÷40=?
2. Have students try to solve 328÷41
3. Exchange ideas in whole class; summarize a consistent method
4. Have students use the vertical way to solve
5. Check

Ms. Chang, a 5th grade teacher with 18 years of experience and associate degree in Chinese
agreed the same methods, but suggested basic practice problems 800÷100, 150÷30 and 240÷60 She also asked questions to remind students to look at the divisor first and estimate it to a close tens number: "what is a divisor close to what tens number?"

Ms. Bao, another 5th grade teacher, with 23 years of teaching and an associate degree in education also emphasized using **algorithms with** review and scaffolding:

1. First, review one digit of divisor, from prior knowledge to knowledge

2. Teach328÷41. View 41 as 40 to try out a quotient. Think about the position of the quotient when trying it and then think about the rhythm of 4 related multiplications.
3. After the success of the try-out quotient, stress on the vertical form of computation.
4. Have students summarize the methods

Mr. Hong, a 4th grade teacher with 28 years of teaching experience and a high school degree summarized his ways:

1. Teaching try-out quotient way
2. Decide the position of the quotient
3. Have students pay attention to the remainder and divisor
4. Have students pay attention to 0 in the quotient
5. The remainder should be less than the divisor

Ms.Ying, a 2nd grade teacher with 17 years of teaching experience and a BA in Chinese addressed her strategies of teaching 328÷41:

1. 328÷41 is the vertical computation of the three-digit number divided by a two-digit number. The key part is how to try out the quotient, which is also the most difficult part.
2. Through appropriate guidance to help students see that 40 is the closest to 41 (based on tens) and can be used as a quotient.
3. The quotient must multiply to 41, not 40.
4. Have students finish the computation independently
5. Recall the example and computation, and summarize the methods of solving 328÷41.

These strategies combined the content and pedagogical knowledge of review & scaffolding, question, and student try, teacher explain, and guide them into a discussion.

(c) Significant Factors between Teachers' Educational Background and Knowledge

Table 7 shows the various relationships between teachers' educational background and knowledge. First, there is no significant correlation between content knowledge and Chinese teachers' educational background; However, there is a significant association between pedagogical knowledge and Chinese grade level of teaching (V = .273, p < 0.002), and there is a strong association between pedagogical knowledge and Chinese teachers' majors at college level (V = .346, p < .0001).

Table 7

The Strength of the Association between Educational Background and Knowledge

	Pedagogy			Content Knowledge		
	Cramer's V	*Eta*	Sig	Cramer's V	*Eta*	Sig
Grade Level	.273	.294	Yes	.148	.133	No
Degree	.242	.274	No	.137	.132	No
Major	.346	.387	Yes	.128	.127	No
Yr. Teaching	.238	2.08	No	.158	.206	No

Table 8 shows that about 50% of Chinese teachers who majored in education would use review and scaffolding compared to 21% of teachers who majored in mathematics; more than 55% of Chinese teachers in math related majors would use different combinations of these approaches.

Table 8

Association between PCK and Major at College: PCK & Major Cross Tabulation

PCK	Major			
	Math	Education	Other	Total
Re & Sca (1)	21.1%	49.7%	34.1%	38.5%
St Tr & Te Ep (2)	8.3%	7.5%	9.1%	7.9%
Question (3)		.5%	2.3%	.6%
Address Miss (4)	.9%	.5%		.6%
1& 2	17.4%	15.5%	11.4%	15.6%
1, 2, 3	19.3%	16.6%	27.3%	18.8%
1& 5	1.8%	1.6%	4.5%	2.1%
2 & 3	4.6%			1.5%

1, 2, 3, 4	2.8%	2.7%	4.5%	2.9%
1, 2, 3, 6	9.2%	.5%	2.3%	3.5%

Table 9 shows that Chinese teachers at lower grade levels tend to use more review and scaffolding (about 45% from grades 1 to 3) than higher grades (around 35% from grades 4 to 6).

Table 9

Association in PCK and Grade Levels of Teaching: PCK & Grade Cross Tabulation

	Grade Levels						
	grade1	grade2	grade3	grade4	grade5	grade6	Total
Re & Sca	44.6%	47.0%	46.2%	33.7%	36.8%	31.1%	39.7%
St Tr & Te Ep	8.9%	9.1%		10.5%	7.4%	11.1%	8.0%
Question		1.5%			1.5%		.5%
Address Mis	1.8%					2.2%	.5%
1& 2	12.5%	10.6%	19.2%	17.4%	13.2%	24.4%	15.8%
1& 4	3.6%			2.3%			1.1%
1, 2, 3	17.9%	22.7%	15.4%	18.6%	17.6%	8.9%	17.4%
2, 3, 4						11.1%	1.3%
1, 2, 3, 4	3.6%	1.5%	9.6%		4.4%		2.9%
1, 2, 3, 6	1.8%	1.5%		5.8%	7.4%	2.2%	3.5%
1, 2, 3, 4, 6	1.8%		1.9%		1.5%		.8%

Discussion

This study investigated Chinese teachers' content and pedagogical knowledge of multi-digit division as well as measures of pedagogical content knowledge and found an association between content and pedagogical knowledge in multidigit division.

Chinese Teachers' Content and Pedagogical Knowledge in Multidigit Division

The results of this study show that to enhance students proficiency in multidigit division, the Chinese teachers used a variety of ways to foster students' sound sense in multidigit division. Notably, about 72% of Chinese teachers prefer to use the estimation method and have their students first try out a quotient with one digit and two-digit number of multiples of 10. This method, supported by NMAP (2008), recommends that teachers should foster a strong sense of numbers that "includes the ability to estimate the results of computations and thereby to estimate orders of magnitude" (p.18) in multidigit division. However, it is difficult for students with instruction and curriculum that frequently overemphasizes routine paper-and pencil calculation for exact answers over other methods like estimation (NRC, 2001; Sowder & Wheeler, 1989). To help students gain the benefits of using estimation, we suggest integrating the estimation skill with other strands rather than teach it as a set of isolated rules (NRC, 2001). The evidence from this study shows that Chinese teachers understand the benefits of estimation and are able to integrate the estimation skill in teaching multi-digit division.

In the US, the algorithms in division require students to determine the exact quotient that has created difficulties for themselves (NRC, 2001). In this study, Chinese teachers showed how to use the estimation method to find the exact quotient without using algorithms. It is very important to note that in this process, Chinese teachers were able to connect student conceptual understanding and adaptive reasoning with using estimation. This connection is supported by the NRC which states that student procedural fluency should be connected with their conceptual understanding and adaptive reasoning. Ms. Ma provided a good example of how to use estimation with understanding:

1. Have students view the divisor 41 as 40 to try out
2. Since using two digits 3 and 2 in the dividend 328 is not enough, so the quotient should be in ones place
3. View 328 as 320 to get a quotient 8
4. $40 \times 8 = 320$.
5. 320 is close to 328. $328 - 320 = 8$. Therefore, $328 \div 41 = 8$

Among Chinese teachers who like to use estimation, 87% of the teachers would like to use a transition method - review and scaffolding - to build a strong connection between prior knowledge and new learning. They would review one digit divisors first, and then review two-digit divisors with multiple of tens. For example, Ms. Pei would like to have students practice

328÷4and 328÷40 first as a transition for solving 328÷41. This common method is called "Dian Di" in China, which means building a solid foundation for new learning. The review and scaffolding is greatly influenced by Confucianism. According to Confucianism, the study should integrate "review" in the learning process (学而时习之). In addition, Confucianism believes that in reviewing prior knowledge, one can always find new

knowledge (Cai & Lai, 1994, An, 2004).

Chinese teachers used the transition of knowledge in this study not only to just teach routine computation, but also to teach for understanding by asking "why" questions in order to promote understanding. Research supports that the foucus of instruction should be on students understanding and explaining (NRC, 2001). About 52 % of Chinese teachers have students try the problems first and then they themselves or their students go on to explain it. For example, Ms. Gao, Mr. Lian, and Ms. Liu in this study would like to create opportunities for students to "discuss, explain, and share their methods." This inquiry process of learning was promoted by Xunzi, a famous scholar in Chinese history: Tell me and I forget. Show me and I remember. Involve me and I understand ("不闻不若闻之，闻之不若见之，见之不若知之，知之不若行之；学至于行之而止矣。"–荀子《儒效篇》).

Measureable Pedagogical Content Knowledge

The results of this study provided insightful way to measure teachers' pedagogical content knowledge through testing the relationship between content and pedagogical knowledge using Cramer's V. This test provided statistical evidence of how a teacher's content knowledge connects to their pedagogical knowledge in three areas of content and six areas of pedagogy. For example, the results of the test show the correlation of 71% between the content knowledge of estimation method and review and scaffolding (PK), 44% of students trying and the teacher explaining (PK), 23% of questioning (PK), 3% of situated learning (PK), and 5 % of addressing misconceptions (PK). With the large effect size, this measurement is very effective. Multiple examples provided vivid evidence of strong pedagogical content knowledge reflected in the association between content and pedagogical knowledge. For example, when using estimation, Mr. Yang used review and scaffolding by teaching 328÷4 and 328÷41 first, then create a situation, and had students try and explain the methods. Other teachers supported students learning by asking questions to promote conceptual understanding, discussing, comparing different methods, and selecting the best methods. The strategies most Chinese teachers used show their strong pedagogical content knowledge as "an integrated knowledge of the development of students' mathematical understanding, and a repertoire of pedagogical practices that take into account the mathematics being taught and students learning it "(NRC, 2001, p.428).

Relationship between Educational and Teaching Backgrounds and Knowledge

The findings shows that Chinese teachers' pedagogical knowledge is strongly related to grade level taught and that their pedagogical content knowledge is strongly related to the majors they chose in college, indicating that more experience in teaching, and more educational courses make teachers more knowledgeable in their efforts to support student learning. The facts of 50% using review and scaffolding in education major compared to 21% of in mathematics major in this study show that the teachers majoring in education understand their students' learning process, one which needs frequent connections to prior knowledge before learning new knowledge. However, the results show that more than 55% of Chinese teachers majoring in mathematics prefer a combination ways of teaching, which means that the teachers with more mathematics courses have more approaches in teaching.
It is interesting to note that there is no significant correlation between content knowledge and Chinese teachers' educational background (about 11% of high school, 57% of associate degree, and 32% of BA degree), which shows that profound content knowledge can be not only learned from the college courses, but can be also enhanced from teaching practice and professional evelopment.

Conclusion

This study shows that teachers' knowledge of multi-digit division has multiple dimensions. It is very important to broaden teachers' knowledge of teaching multi-digit division beyond simply using algorithms. Using estimation is a good approach as a bridge to foster students' sound number sense and fluency in computation, thereby supporting effective and efficient student learning.

The measures of teachers' pedagogical content knowledge with the indication of a statistical strength of association between content and pedagogical knowledge in this study provided a new, rigorous, and empirical approach to measuring teacher knowledge. Further studies are needed to test teachers' pedagogical content knowledge in different content areas and different countries in order to validate this approach in a broader range.

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