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Conceptualizing and Cultivating Mathematical Practices in School Classrooms

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This paper elaborates on the nature and characteristics of mathematical practices in school classrooms. A review of existing literature reveals the multifaceted nature of mathematical practices, as (1) productive mathematical activities, (2) expertise and proficiency in mathematics, and (3) mathematical thinking habits. A conceptual framework is established to highlight the fundamental features of mathematical practices to be cultivated among learners: (1) engagement and commitment, (2) development and employment of knowledge, skills, and strategies, and (3) internalization and habitualization. The paper further discusses the implications of this conceptualization for implementing the Standards for Mathematical Practice.

Key words: mathematical practices, mathematical habits of mind, Common Core State Standards in Mathematics, Standards for Mathematical Practice

Leadership in mathematics education in the United State has paid close attention to the importance of *mathematical practices* in the past decade. The RAND Mathematics Study Panel (2003) identified mathematical practices as one of the three focus areas of a long term, strategic research and development program in mathematics education. Such an emphasis was rooted in the panel's concerns that (1) mathematical practices are often left implicit in instructions, and ineffective implementation of instructions that are supposed to facilitate these practices has led to contentious debates over the impact of standards-based reform curricula; (2) many questions about mathematical practices remain unanswered, and the lack of adequate knowledge about these practices has also caused controversy over mathematics education improvement efforts. The panel believed that a better understanding of the nature of these practices and how they are attained could contribute greatly to improving student learning.

The Common Core State Standards for Mathematics (CCSSM) (Common Core State Standards Initiative, 2010) reaches a new milestone by introducing the *Standards for Mathematical Practice* as overarching principles for mathematics learning and teaching. For each of the eight mathematical practices, the standards provide a one-paragraph description of the various types of student activities and desired learning outcomes that signify the practice. Altogether these descriptions lay the groundwork for teachers to learn and understand the fundamental practices. However, considering the complexity in mathematics teaching and potential variances in teachers' interpretations and implementations of standard-based curricula (Ball, 1996; Lloyd, 1999; Schoenfeld, 2002; Senk & Thompson, 2003), much more detailed explanations and exemplifications of mathematical practices are needed for teachers to align their classroom instructions with the original intents of the standards. This is also part of the call for mathematics teacher preparation and professional development programs to develop practice-focused curriculum and training (Ball & Forzani, 2011) that would help teachers to interpret and implement the CCSSM in comprehensive and coherent ways (National Council of Teachers of Mathematics, National Council of Supervisors of Mathematics, Association of State Supervisors of Mathematics, and Association of Mathematics Teacher Educators, 2010).

This paper attempts to gain deeper insights into mathematical practices by theoretically addressing two related research questions:

1. What is the nature of mathematical practices?
2. Which characteristics of mathematical practices should be emphasized and developed in school mathematics classrooms?

The paper first reviews how existing documents characterize mathematical practices and its twin notion, *mathematical habits of mind*, then builds a conceptual framework that identifies the key aspects of mathematical practices to be prioritized and measured in mathematics learning experiences. From there, the paper discusses the implications of such a conceptualization for cultivating and assessing mathematical practices in school classrooms, as well as for mathematics teacher education research and practice.

The Duality of Mathematical Practices

The notion of mathematical practices didn't originate broadly to refer to everything that everyone does in using or learning mathematics, but denoted specifically those skillful activities and productive strategies undertaken routinely by experts in mathematics. For instance, the RAND Mathematics Study Panel (2003) first defined mathematical practices as "mathematical activities in which mathematically proficient people engage as they structure and accomplish mathematical tasks" (p. 11), later rephrased it as "mathematical know-how – what successful mathematicians and mathematics users *do*" (p. 29). However, since mathematical practices can be taken as an indicator of expertise or competency in mathematics, they shouldn't be limited only to mathematical experts. As the panel claimed, they should become a key element of one's mathematical proficiency which can be developed or learned over time.

The CCSSM introduces the eight mathematical practices as "varieties of expertise that mathematics educators at all levels should seek to develop in their students", then points out that these practices "rest on important 'processes and proficiencies' with longstanding importance in mathematics education" (p. 6).

For “processes”, the CCSSM makes direct reference to the five *process standards* in Principles and Standards for School Mathematics (NCTM, 2000): problem solving, reasoning and proof, communication, connections, and representation. According to NCTM (2000), these five standards “highlight ways of acquiring and using content knowledge”. Together with the five *content standards*, they “describe a connected body of mathematical understandings and competencies -- a comprehensive foundation recommended for all students” (p. 29).

Meanwhile, the CCSSM links “proficiencies” to the five strands of mathematical proficiency illustrated in National Research Council’s report, Adding It Up: Helping Children Learn Mathematics (Kilpatrick, Swafford, & Findell, 2001): adaptive reasoning, strategic competence, conceptual understanding, procedural fluency, and productive disposition. These are the most essential knowledge, skills, abilities, and beliefs that the authors see as indicators of success in mathematics learning.

When explaining each mathematical practice, the CCSSM specifies what “mathematical proficient students” are inclined to do and able to do in learning, both in general and specific to a grade band.

Through the RAND Mathematics Study Panel report and the practice standards in the CCSSM, mathematical practices are transformed from what mathematics experts do and how they think into an important goal for student learning and understanding in school mathematics.

Across the above descriptions and explanations there emerge two common features of expected mathematical practices in the context of school classrooms: (1) mathematical practices are productive activities and thinking processes that learners competently engage themselves in, and through which learners’ knowledge, skills, and strategies in mathematics are developed and applied; (2) Being able to undertake mathematical practices is a fundamental type of expertise, proficiency, and capability that should be cultivated among all mathematics learners. Hence learners’ mathematical practices exhibit a process-product duality: they make up both a means to and an end of mathematics learning and understanding.

Mathematical Practices as Habits of Mind

According to the online dictionary provided by Dictionary.com, the word “practice” as a noun has the following five basic explanations:

1. habitual or customary performance; operation
2. habit; custom
3. repeated performance or systematic exercise for the purpose of acquiring skill or proficiency
4. condition arrived at by experience or exercise
5. the action or process of performing or doing something

Explanations No. 5 (“action or process”) and No. 4 (“condition”) are the broadest interpretations of practice, and seem well match the process-product duality of mathematical practices revealed earlier. In contrast, the first two interpretations are most specific and indicate that certain types of practice would go beyond single or repeated processes of “doing something” to become internalized habits or customs.

In terms of mathematical practices, since they are characteristic of the activities that mathematically proficient people carry out on a routine basis, and also signify the kind of expertise these people demonstrate and utilized most frequently, it would be reasonable to believe they are intrinsic “habits” or “inclinations” in doing mathematics, rather than actions based on random decisions, coincidental inspirations, or external instructions.

Some mathematics education researchers implicitly acknowledged such a habitual nature of mathematical practices. For instance, Schoenfeld (1992) identified mathematical practices as one of the five fundamental aspects of *thinking mathematically*. He then quoted Resnick’s (1989, p. 58) comment to further examine the nature and role of mathematical practices: “Becoming a good mathematical problem solver – becoming a good thinker in any domain – may be as much a matter of acquiring the habits and dispositions of interpretation and sense-making as of acquiring any particular set of skills, strategies, or knowledge.” Such a quotation suggests that Schoenfeld subconsciously considered “habits and dispositions” as a defining property of mathematical practices.

A few other researchers linked mathematical practices to the notion of *mathematical habits of mind*. For example, Selden and Selden (2005) used the two phrases interchangeably, Bass (2008) considered mathematical habits of mind as practices – things that mathematicians do, whereas Smith (2011) also treated these two notions as congruent.

Back in the late 1980s, two of the most influential documents for mathematics education reform, *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989), and *Everybody Counts* (National Research Council, 1989), both pointed out the importance of mathematical habits of mind to developing mathematical literacy among all learners in an information society. However, both documents seemed to take the phrase habits of mind as self-explanatory without elaborating on its meaning.

Cuoco, Goldenberg, and Mark (1996) gave one of the first systematic and in-depth illustrations of habits of mind and their role in school mathematics. The authors pointed out that school mathematics was typically studied as a collection of established results and methods, and had little to do with how mathematics was created and applied outside of school. They believed mathematics curriculum reform should go beyond simply replacing old content with modern results, and it was more important for students to learn “the habits of mind used by the people who created those results” (pp.

375-376). They proposed to turn the priorities around, using habits of mind as an organizing principle for mathematics curricula.

The authors interpreted “habits of mind” by rephrasing it in several other ways, such as “the methods by which mathematics is created”, “the techniques used by researchers”, and “the ways that mathematicians *think* about problems” (p. 376). Further, they provided examples of four types of habits of mind that are important in mathematics: (1) general habits of mind across all disciplines; (2) habits of mind mostly common in mathematics; (3) geometric ways of thinking; (4) algebraic ways of thinking.

Since then, a few new publications have continued to promote mathematical habits of mind (Achieve, 2008; Ball, 2005, 2008; NCTM, 2000) or similar notions, such as *habits and values of mathematicians* (Seaman & Szydlik, 2007) and *mathematical ways of thinking* (Cuoco, 1998; Harel & Sowder, 2005). By 2010, mathematical habits of mind was no longer a topic merely for mathematics education research or mentioned only rhetorically in standards documents. Cuoco, Goldenberg, and Mark (2010) and Mark, Cuoco, Goldenberg, and Sword (2010) provided customized lists of habits and examples for school mathematics teachers at different grade levels. These lists echoed and actually overlapped with the mathematical practices in both the draft and the official versions of the CCSSM, as well as with the *reasoning habits* in the *Focus in High School Mathematics – Reasoning and Sense Making* (NCTM, 2009). NCTM (2009) defined a reasoning habit as “a product way of thinking that becomes common in the process of mathematical inquiry” (p. 9), which resembles some of the descriptions of mathematical practices and habits of mind discussed earlier.

Several researchers explicitly addressed the habitual and inclinational characteristics of mathematical habits of mind. Goldenberg (1996) described mathematical habits of mind as attributes that “one acquires so well, makes so natural, and incorporates so fully into one's repertoire, that they become mental habits -- one not only can draw upon them easily, but one is likely to do so” (p. 13). Leikin (2007) stated that “employing habits of mind means inclination and ability to choose effective patterns of intellectual behavior” (p. 2333). In summarizing existing conceptualizations of mathematical habits of mind, Lim and Selden (2009) and Selden and Lim (2010) distinguished *the habituated characteristic* from *the thinking characteristic* (habits of mind as internalized ways of thinking). Among others, they cited Mason and Spencer’s (1999) notion of *knowing-to act in the moment* as an illustration of the habituated characteristic: it is a tacit knowledge that depends on the context or problem situation and becomes present when it is needed.

A Conceptual Framework

Reviews in the previous two sections partially addressed the first research question (“What is the nature of mathematical practices?”) and revealed that mathematical practices are productive and habitual ways of thinking and reasoning in mathematics. Mathematical practices in their fully developed status demonstrate a multifaceted nature: they are (1) productive mathematical activities, (2) expertise and proficiencies in mathematics, and (3) mathematical thinking habits. These features were initially observed from the intellectual behaviors of mathematically sophisticated people, and have been increasingly expected to develop among mathematics learners in school classrooms.

The current section targets the second research question ("Which aspects of mathematical practices should be focused on in school mathematics classrooms?"). Outlined below is a conceptual framework that identifies and describes three overarching themes underlying all specific types of mathematical practices expected of student learning. These themes match and reflect the three features of mathematical practices summarized above. The framework also suggests basic ways of evaluating each theme in mathematics learning and teaching.

Behavioral engagement and commitment. The development of mathematical practices as productive intellectual endeavors requires learners to actively work on mathematical tasks and activities, individually or collaboratively. They analyze problem situations, engage in thinking and reasoning processes, make conjectures and argumentations, discuss with peers, classmates, and teacher, utilize technology and other resources, carry out numerical computations, algebraic manipulations, and physical actions such as taking mathematical and scientific measures of real world objects and processes, sketching geometric shapes and coordinate graphs. They constantly reflect on the ongoing progress and strategies against their goals and available resources, and make necessary adjustments for improvement. They are committed and persistent until the completion of the tasks and resolution of the problems.

This aspect of mathematical practices could be evaluated in terms of the levels of learners' engagement and commitment (high, moderate, low, etc.). A high level of engagement is a necessary, although not sufficient, condition for productive mathematical practices.

Employment and development of knowledge, skills, and strategies. Mathematical practices as a type of expertise or proficiency in mathematics involves various kinds of knowledge, skills, tools, and strategies, including informal and intuitive knowledge, knowledge of facts, definitions, and properties, knowledge of algorithms, formulas, and other routine procedures, the skills needed in carrying out these procedures, basic logical reasoning skills, proof and justification techniques, and problem solving strategies. During engagement in mathematical practices, learners have to first apply previously acquired knowledge, skills, and strategies in modeling and problem solving. Such a process could in turn help learners to strengthen their conceptual understanding, procedural fluency, reasoning skills, and problem solving strategies. Further, learner may likely develop and consolidate new knowledge and skills through long term engagement in mathematical practices.

Correspondingly, evaluating this aspect of mathematical practices could go in three main directions: (1) status analysis - examining *what* knowledge, skills, and strategies are used, and the *quality* of these knowledge, skills, and strategies; (2) use and effect analysis - examining *how* knowledge, skills, and strategies are used, *how effectively* they are used, and whether they lead to *optimal solutions* to the problem situation; (3) growth analysis - examining *whether and how* prior knowledge, skills, and strategies evolve during practices, *what* new knowledge, skills, and strategies are developed, *how* and *how effectively* they are developed.

Internalization and habitualization. Learners start experiencing certain types or aspects of mathematical practices by engaging in teacher-designed learning activities or by mimicking the kinds of practices their mathematics teacher demonstrate in teaching. Ultimately, however, learners should be expected to fully make sense of and appreciate the behaviors, methods, and strategies that are characteristics of the mathematical practices, so that these become internalized and habitual to the learners. When facing a particular mathematical task or problem situation, learners with a high level of habitualization would spontaneously choose and launch those mathematical practices that are most relevant to the task or problem situation.

In terms of evaluating the habitual features of learners' mathematical practices in classrooms, the framework proposed by Costa and Kallick (2000) on general habits of mind sheds light on what could or should be measured. The framework encompasses a few intertwined psychological traits, including (1) Sensitivity: being able to recognize situations and opportunities for employing certain habits; (2) Inclination: feeling the tendency toward employing certain habits; (3) Value: choosing to employ certain types of habits rather than others. When the phrase "employing habits" is replaced by "engaging in practices", these three traits could form a set of lenses for evaluating the habitual aspects of mathematical practices and their growths over time.

Discussion

Based on the reviews conducted in the first two sections and the framework introduced in the third section, this current section further discusses the two research questions from student learning, assessment, and mathematics teacher education perspectives.

Practices are Beyond Behavioral

One of the challenges in mathematics education has to do with the fact that many popular catch phrases do not have rigorous or universally accepted definitions therefore they are often subject to different interpretations and uses. A major consequence of such a situation could be that "the rhetoric…… has been seen more frequently than the substance" (Schoenfeld 1992). New goals or ideas in mathematics education may be implemented in fashionable but superficial ways such that the original intention gets lost. Two of the most classic examples are *conceptual understanding* (Hiebert, 1986) and *problem solving* (Schoenfeld, 1992). This is more or less the case for the notion of mathematical practices. In its broadest sense, practices are actions or activities with certain purposes. Narrowly speaking, practices are physical behaviors. Neither of these two accurately captures the nature of mathematical practices. Mathematical practices are beyond simply "doing something" in or with mathematics. They should be activities that are skilled, productive, and habitual, or at least incorporate these features as expected outcomes. In the meantime, mathematical practices do not have to be always physical or visible, they can be mental thinking and reasoning activities that are quiet and motionless.

Some curriculum materials and documents promote active learning in mathematics by showing pictures of students working collaboratively and using hands-on manipulatives or technologies. Despite all of its positive intentions and influences, this kind of portrait might have reinforced a stereotypical or superficial concept of mathematical practices. Although physically engaging into activities and collaborations is a crucial aspect of mathematical practices, it can hardly be separated from mental processes. Further, merely engaging in physical and mental activities is far from sufficient for mathematical practices to be accomplished. Students could be "busy doing stuff", "working in group", or "having lots of fun", but none of these is a substantial indicator of productive mathematical practices. In designing and implementing learning activities that promote mathematical practices, teachers should look beyond the behavioral involvement of students (the first theme in the conceptual framework) and pay close attention to the cognitive aspects: the application and development of knowledge, skills, and strategies, and the internalization and habitualization of mathematical behaviors (the second and third themes in the framework). Same thing can be said for mathematics teacher educators and researchers who are to observe or evaluate the status, quality, effectiveness, and development of mathematical practices in student learning.

Evaluation and Assessment

Learners' mathematical practices can be measured in two basic approaches. One is to observe classroom teaching and learning and evaluate the status and growth of mathematical practices. In the previous section, the description of the conceptual framework already suggests ways of evaluating each of the three themes of mathematical practices. Such an evaluation can be conducted by mathematics teacher educators or researchers, or by teachers themselves, for individual types of mathematical practices, or across all eight practices proposed by the CCSSM.

The other approach is to formally assess learners' mathematical practices as part of an educational accountability system. This wasn't possible before the release of the CCSSM since mathematical practices were not a required assessment topic for schools, nor were they clearly defined most of the time. Now with the Standards for Mathematical Practice soon to be implemented, developing guidelines and instruments for assessing mathematical practices becomes fully necessary, otherwise, as EDC (2011) stated,

> *Any system that rewards and/or punishes teachers and schools on the basis of measured results can expect that anything that isn't reflected in the measuring instruments will be ignored completely. The Mathematical Practices will be taken seriously in curriculum and teaching if, and only if, they are taken seriously in testing. It can be expected, then, that the developers of the CCSS, and the States that collaborated in calling for the development of the CCSS, will work with the developers of assessments to ensure that the Mathematical Practices are taken seriously in testing.*

The Smarter Balanced Assessment Consortium was in fact formed to develop assessment instruments along with the implementation of the CCSSM. Most recently, the Consortium established major goals of student learning outcomes and made statements about the kinds of evidence that would support each goal. These specifications will be used by item writers and reviewers to create high quality assessment items.

Teachers and Mathematical Practices

Mathematical practices are productive and habitual ways of thinking and doing mathematics. They are best acquired by learners through their immersion in mathematical exploration and problem solving activities, rather than learned as a list of specific objects (Levasseur & Cuoco, 2003). Correspondingly, mathematics teachers themselves should first have the opportunities to experience, internalize, and exercise mathematical practices in their own learning and use of mathematics (Conference Board of the Mathematical Sciences, 2001; Lesh & Doerr, 2003). By doing so they are capable of in turn model those practices and habits for learners, and design and engage students in authentic mathematical activities that would naturally foster those practices and habits (Cohen, 2011; Levasseur & Cuoco, 2003; Stevens, 2001; Stevens, Matsuura, Rosenberg, & Sword, 2007).

Once internalized, mathematical practices could likely become an inherent component of a mathematics teacher's professional knowledge and expertise that are useful for teaching. This relates mathematical practices to the notion of *mathematical knowledge for teaching* (MKT) (Ball, 1999, 2000; Ball & Bass, 2000, 2003b; Ball, Lubienski, & Mewborn, 2001; Cuoco, 2001; Davis & Simmt, 2006). In some more recent publications, researchers (Ball & Bass, 2009; Ball, Thames, & Phelps, 2008) introduced a new subcategory of MKT, *knowledge of the mathematical horizon*, and characterized it as "an awareness of the large mathematical topics landscape in which the present experience and instruction is situated" (Ball & Bass, 2009, p. 5). Such horizon knowledge is supposed to be shared among all mathematicians and mathematics teachers. According to these researchers' further illustration, knowledge of the mathematical horizon includes "key mathematical practices" as one of four constituent elements.

Hence, mathematical practices are not only expected of school students, but also of mathematics teachers who intend to foster mathematical practices in his or her classroom. In facilitating the implementation of the mathematical practice standards in CCSSM, mathematics teacher preparation and professional development programs bear dual responsibilities: (1) cultivating mathematical practices among teachers, as experiences and/or as one type of MKT; (2) supporting teachers in fostering mathematical practices among their students.

Future Directions

The conceptual framework established in this paper intends to provide mathematics teacher educators and education researchers with a potentially useful lens for observing the status and evaluating the growths of mathematical practices among school mathematics learners. More detailed descriptions and instruments are needed for measuring the following: (1) each of the three themes of the practice; (2) the growth of mathematical practices over time; (3) the productivity of a mathematical practice; (3) factors that influence the productivity of math practices.

Mathematical practices should be further evaluated or examined, as a special type of knowledge of the mathematical horizon, or MKT. Although MKT has been discussed for over a decade, the notion of horizon knowledge is relatively new and needs to be more thoroughly examined, both theoretically and empirically (Ball & Bass, 2009; Foster, 2011; Zazkis & Mamolo, 2011).

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