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Mathematical Knowledge for Teaching Algebraic Routines: A Case Study of Solving Quadratic Equations

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This paper describes the sequence of mathematical practices that a teacher designed and enacted during three consecutive lessons about four algebraic routines for solving quadratic equations, and focuses on the mathematical knowledge that is entailed in the teacher's actions and decisions. Among the three domains of mathematical knowledge for teaching (MKT), the teacher's knowledge of the equation solving routines played the most crucial role in shaping the lessons and potentially promoting mathematical proficiency in a balanced approach. The findings suggest that mathematics teacher preparation and professional development programs should provide more opportunities for teachers to revisit in-depth the features and applicability of various mathematical routines, and develop skills in making instructional decisions that would balance all domains of teachers' MKT, teacher beliefs, and other factors related to proficiency in algebraic routines.

Key words: mathematical knowledge for teaching, algebraic routines, quadratic equations, mathematical proficiency

The last decade has witnessed mathematics education researchers' tremendous interests and efforts in conceptualizing and assessing the kinds of mathematical knowledge that teachers draw upon or need to acquire for effective teaching (Adler & Davis, 2006; Ball, 1999, 2000; Ball & Bass, 2000, 2003; Ball, Bass, Hill, & Schilling, 2005; Ball, Lubienski, & Mewborn, 2001; Ball, Thames, & Phelps, 2008; Cuoco, 2001; Davis & Simmt, 2006; Ferrini-Mundy & Findell, 2001; Hill, Ball, & Schilling, 2004, 2008; Ma, 1999; Moreira & David, 2008; Stylianides & Ball, 2008). The entire body of such knowledge has been typically phrased as *mathematical knowledge for teaching* (MKT). A number of the systematic explorations focused specifically on MKT in school algebra (Artigue, Assude, Grugeon, & Lenfant, 2001; Ferrini-Mundy, Burrill, Floden, & Sandow, 2003; Ferrini-Mundy, McCrory, & Senk, 2006; Heid, 2007), and proposed theoretical frameworks and developed preliminary assessment instruments. Consider the foundational role and high-stake status of algebra in the school mathematics curriculum. However, more research is needed for researchers and mathematics teacher educators to deepen their understanding and design more robust assessments of MKT in regard to school algebra.

Because modern school algebra is a complex and evolving entity that involves multiple, and often competing, conceptions, focuses, and approaches (Chazan, 2000, 2008; Kieran, 2007; Kilpatrick & Izsák, 2008), research on algebra-related MKT would have to go beyond the overarching features of algebra and address the unique characteristics and dynamics of individual algebraic topics, strands, or types of activities. In recent years the researcher has focused his work on a particular content theme in algebra – basic routines and procedures, which include the algebraic rules, algorithms, and formulas that can be applied to given inputs and yield desired outcomes through finite steps (e.g., the distributive property of multiplication over addition and its extensions, such as the so-called FOIL formula $(a + b)(c + d) = ac + ad + bc + bd$ and identities $(a + b)^2 = a^2 + 2ab + b^2$ and $(a + b)(a - b) = a^2 - b^2$; various established methods or formulas for solving linear and quadratic equations, such as the balancing and backtracking methods, factoring, completing the square, and the quadratic formula).

Algebraic routines and procedures are closely tied to the five interwoven essential strands of mathematical proficiency (Kilpatrick, Swafford, & Findell, 2001), in particular, conceptual understanding, procedural fluency, and strategic competence. Ideally, these strands are interdependent and are to be developed simultaneously in balanced ways. In practice, however, there lacks a consensus among mathematics educators on the distinctions and relationships between conceptual and procedural knowledge (De Jong & Ferguson-Hessler, 1996; Hiebert, 1986; Li, 2007; Star, 2000, 2005). Consequently, the twin goals of developing procedural fluency and conceptual understanding are often viewed as competing for attention and priority in school mathematics (Kilpatrick, Swafford, & Findell, 2001), which in turn creates a false dichotomy (Wu, 1999). Mathematics teachers with distinct beliefs and MKT regarding these strands may likely approach algebraic routines with different goals and teaching strategies.

Hence, studying the kinds of MKT used in teaching algebraic routines would help researchers and teacher educators to evaluate the depth and breadth of individual mathematics teachers' own conceptions of the routines and best ways of teaching and learning them, and make better sense of teachers' instructional preferences, design, reasoning, and decision-making. Such studies would also potentially identify the most crucial features of MKT that could support effective teaching and facilitate balanced and substantial growth in students' algebraic proficiencies, including the most recent emphasis on developing reasoning and sense making skills (Graham, Cuoco, & Zimmermann, 2010; National Council of Teachers of Mathematics, 2010).

Research Objective and Question

In previous studies (Li, 2007, 2008, 2009), the researcher measured secondary school mathematics teachers' MKT in algebraic equation solving based on teacher participants' written responses to an assessment instrument and data generated from follow-up interviews. In a current study on MKT, the researcher looked into an individual mathematics teacher's daily instructional activities in actual school and classroom settings to search for traces and indications of MKT. Through this case study, the researcher expected to generate richer, more subtle, and more authentic details of individual teachers' MKT that is demonstrated in and has influence on the teachers' pedagogical routines reasoning in teaching algebraic routines.

Specifically, the researcher chose quadratic equation solving as the algebraic routine to focus on, and attempted to answer the following questions: What MKT is entailed in a mathematics teacher's instructional design and enactment, when the teacher is teaching an algebraic routine (such as solving quadratic equations) and aims to develop multiple strands of mathematical proficiency simultaneously among students?

Conceptual Framework

In building a tentative framework for MKT for his own studies, the researcher compared a variety of existing theoretical perspectives on MKT, including the most influential ones proposed by Shulman (1986a, 1986b, 1987), Ball, Thames, and Phelps (2008), and Thames and Ball (2010). Three common core domains of knowledge emerged during this process, which resemble very much the three dimensions of teachers' professional knowledge for teaching algebra, *epistemological*, *cognitive*, and *didactic* (Artigue, Assule, Grugeon, & Lenfant, 2001), and the three components of teachers' mathematical knowledge base, *mathematics content, student epistemology*, and *pedagogy* (Harel & Lim, 2004). These three domains were adapted by the researcher to establish his own framework for MKT, and they were described as follows:

1. Knowledge of the mathematical subject matter: teachers' knowledge of the mathematical content in its subject matter forms (interrelated concepts, processes, properties, methods, and sequences of reasoning), both as an academic discipline and as a course of study.

2. Knowledge of pedagogical representations: teachers' knowledge of the pedagogical form of the mathematical content, as presented by instructional media (e.g., textbooks, manipulatives, visual aids, and electronic technologies) and through various teaching strategies. Such content is the intermediate product when the subject matter is being unpacked, linked, reorganized, and tailored by the teachers through particular sequences of curriculum units and topics, as well as the uses of various examples, metaphors, models, explanations, questions, tasks, tools, and technologies, etc.

3. Knowledge of learners' conceptions: teachers' knowledge of the cognitive forms of the mathematical content, i.e., learners' mental representations of the mathematics subject matter, which results from learners' interactions with the subject matter, the teacher, other learners, instructional technologies, and other factors in the learning environment.

The researcher believes that these domains of knowledge would evolve along with the growth in teachers' mathematics learning experiences and teaching expertise, in teacher preparation and professional development programs, and in their daily teaching practices. These domains of knowledge would be cognitive entities existing in teachers' mental registry. They have major impact on, and could also be reflected by, teachers' instructional design and decision making. Meanwhile, these knowledge domains would likely be constantly modified or refined during teachers' daily interactions with curricular content, learners, and school context. Such beliefs have underlain the entire research design, implementation, and analysis processes.

Methods

Context and Participant

The study took place in one of the largest and most diverse school districts in California. The teacher who participated in the case study, Mary (pseudonym), teaches mathematics at an urban high school in the district. According to the latest profile, student population in the school includes 43.5% Hispanic, 30.7 % white, 12.5 % African-American, and 9.5% Asian. Over 17 primary languages are spoken in the homes of the students. Nearly half of the students are on a free or reduced lunch program.

At the time of the study, Mary had taught high school mathematics for 8 years, and was the leader of the mathematics department in her school. She was chosen for this study because of her experiences and success in working with ethnically and economically diverse students to learn mathematics.

Data Collection

In the Fall of 2010, the researcher observed three consecutive 90-minute lessons taught by Mary in her 9th grade algebra class, which will be named Lessons 1, 2, and 3, respectively, from the rest of this paper. In these lessons, Mary introduced the quadratic formula for solving quadratic equations, addressed various aspects of this method, and compared it to other methods.

Four main sources of data were collected in Mary's class: (1) teaching videos of the three lessons; (2) classroom observation notes; (3) artifacts used by Mary or her students, including lesson plans, presentation slides and transparencies, handouts, worksheets, and student work, etc.; and (4) semi-structured interviews conducted with Mary and audiotaped right after each lesson.

Prior to the data collection process, human subject research applications were approved by the Institutional Research Board at the researcher's institution and by the school district. Informed consent forms were signed by and collected from Mary and the parents or guardians of all students in Mary's class.

Data Analysis

During data analysis, the classroom teaching videos and the interview audiotapes were first transcribed. Both the transcripts and the observation notes then underwent two rounds of coding. The first round focused on the various pedagogical moves that Mary made in the three lessons surrounding the methods for solving quadratic equations. The rationale for this initial analysis is that teachers' actions in response to teaching tasks are places where teachers' knowledge in action can be identified (Marcus & Chazan, 2010). Codes were categorized and compiled into themes such using *representations*, making *connections* and making *justifications*, which overlap with the five mathematical processes addressed by NCTM (2000). The researcher named these themes as *mathematical practices* since they are in essence very similar to those mathematical practices defined in by the Common Core State Standards Initiative (2010). They are physical or mental actions that can be carried out by both teachers and students. As pedagogical moves, they are more closely tied to the focal mathematics topic (in this case, routines for solving quadratic equations) than those general instructional activities or strategies such as selecting tasks, asking questions, orchestrating discussions, or coordinating group work.

The second round of coding was based on the conceptual framework for MKT presented earlier. The researcher first related each of the mathematical practices identified in the first round of coding to one of the three domains of knowledge in the framework, then unpacked and coded the specific MKT entailed in each of the practices. Afterward the researcher combed through all transcripts and notes searching for other evidence and indications of MKT. In the last step of data analysis, the researcher examined all artifacts to triangulate with the themes which emerged from the observation and interview data.

Results and Findings

Mathematical Practices Staged in the Lessons

Throughout the three lessons, Mary had engaged her class into seven mathematical practices which are mostly sequential:

Representations. In Lesson 1, Mary introduced the quadratic formula to her class by employing four kinds of representations in a row: symbolic, verbal, rhythmic, and metaphorical. She first presented the formula for the general form of quadratic equations, $ax^2 + bx + c = 0$, then asked a student to read the formula aloud, and asked the entire class to sing the formula as a rhythm: "*x* equals negative *b*, plus or minus the square root, b squared minus 4*ac*, all over 2*a*.". She also told the story of *The Sad Little Bee* three times as a phonetic cue:

> *The bee is sad (negative), and he is feeling wishy-washy, maybe he will go or maybe he won't (plus or minus). It's about going to the radical party. He's feeling a little squared, about the four awesome cheerleaders. The entire party was over, however, by 2am.*

Exemplifications. The above four representations were followed immediately by five examples of solving quadratic equations through which Mary demonstrated in detail how to identify the coefficients *a*, *b*, and *c*, substitute them into the formula, and carry out the numerical computations. These examples were well selected to cover several different situations: (1) the leading coefficient *a* equals 1 versus (2) *a* is a whole number greater than 1, and (3) the linear coefficient *b* is zero versus (4) *b* is non-zero.

In the first 55 minutes of Lesson 3, Mary led the class through a second and more comprehensive round of exemplifications, going over seven different equations and demonstrated in detail how one or more of the four basic methods (factoring, complete the square, the quadratic formula, and graphing) are used to solve each of these equations. These practices prepared the class for discussion on the features of the four methods at the very end.

Derivation. In Lesson 2, Mary spent nearly one hour out of the total 90 minutes on the derivation and justification of the quadratic formula. She provided the students with a handout that compares side-by-side the 14 parallel steps of the completing the square procedure when it was applied to two problem situations: (1) solving the specific quadratic equation $3x^2 + 7x + 2 = 0$, and (2) transforming the general quadratic equation $ax^2 + bx + c = 0$ step-by-step to induce the quadratic formula.

Justifications. To make sure her students were familiar with and understood the derivation of the quadratic formula, Mary went further to ask the class to justify each of the 14 demonstrated steps. As a preparation, she first let the students turn to the back of the handout which showed six mathematical properties and processes (complete the square, factor, subtraction property of equality, find common denominators, division property of equality, and taking the square root of both sides), and use these to explain six steps selected among the 14. Afterward, she asked the class to turn back to the side with the 14 parallel steps and justify the rationale of each step.

Connections. The derivation and justification activities provided an opportunity through which Mary's students could realize the relationship and connections between the quadratic formula and other foundational properties and methods, such as completing the square. She didn't require the students to reconstruct the derivation process on their own, but did expect them to understand the fact that, despite the seemingly different outlooks and levels of sophistication of the two procedures (numerical vs. symbolic), solving specific quadratic equations by completing the square and deriving the general quadratic formula actually follow the exact same procedure and both are based on the same set of mathematical properties and processes.

Applications. During the last 15 minutes of Lesson 1 and the last 20 minutes of Lesson 2, Mary asked her students to work in pairs and use all four methods (factoring, complete the square, the quadratic formula, and graphing) to solve assigned quadratic equations. Her goal appeared to be that her students not only became fluent with each of these methods but also understood them better in terms of their similarities and differences. Together with the comprehensive exemplifications during the first hour in Lesson 3, these applications and contrasts that occurred across two lessons laid the foundation for the class' final discussion on the features of the four methods.

Comparisons. In the last 35 minutes of Lesson 3, Mary initiated a whole class discussion on the strengths and limitations of the four methods. After eliciting preferences and comparisons from the class, she summarized the typical situations in which one of these methods would work better: (1) use graphing if the graph of the trinomial is already given; (2) use factoring if the lead coefficient *a* equals 1 and the trinomial is "composite" (reducible); (3) use completing the square if *a* is 1 and the trinomial is "prime" (irreducible); and (4) use the quadratic formula if *a* is not 1 and the trinomial is prime. Right afterward she went through three specific examples with her students.

Her goal here was to provide the students with some general guidelines on choosing the best method for solving a given quadratic equation. She briefly touched on two of the five features of mathematics algorithms (Kilpatrick, Swafford, & Findell, 2001): (1) generality, e.g., the quadratic formula works for any quadratic equation while factoring doesn't if integral coefficients are desired in the linear factors; and (2) efficiency, e.g., the factoring method would typically work faster than completing the square.

MKT Entailed in the Mathematical Practices

The mathematical practices identified in Mary's three lessons provided an important venue for the three domains of mathematical knowledge she utilized to surface. Her conversations with the researcher during the post-observation interviews offered further information on her MKT.

Knowledge of the mathematical subject matter. Mary's knowledge of the subject matter was reflected mainly through five of the mathematical practices she enacted: application, derivation, justification, connection, and comparison. During the practices and the interviews, Mary demonstrated a solid understanding of the quadratic formula in terms of the following three aspects:

1. She was able to apply the formula competently to solving various kinds of quadratic equations. She understood that one has to write a given equation into the standard form before substituting the coefficients in the formula, and to correctly perform the numerical computations one needs to follow the order of operations and apply basic properties of fractions and radicals.

2. She was very familiar with the derivation process of the formula, including the underlying mathematical property of each step, as well as the connections between the formula and other methods for solving quadratic equations.

3. She knows clearly about some of the basic features of the quadratic formula (e.g., generality or applicability, accuracy, and efficiency) in comparison with other methods. She understood that the quadratic formula is most general and applicable in solving quadratic equations, and in turn shared this knowledge to her students:

> *It works on everything, under any conditions, and makes it easy and beautiful. It's fabulous. At the end of the day it would solve everything.*

Meanwhile, she understood and paid attention to the limitations of the other three methods. She believed it is important to convey these to her students, as an alternative way to convince them the importance of the quadratic formula:

> *There are actually some really great reasons to memorize this [formula]... If I'm gonna graph a quadratic equation and find the roots, it's a lot of work, and often times the roots are not integers, the graphing won't land on pretty numbers, they land on yucky numbers, we can't read on the graph……*
>
> *Sometimes factoring is just a pain. It's easy if they are factorable, but what if they are prime? You can't even use this [factoring method]……*
>
> *And then completing the square is a bit grove, the problem with completing the square is when the first coefficient is not 1. If the coefficient A is not 1, then completing the square is a pain.*

Knowledge of pedagogical representations. Mary's knowledge of pedagogical representations was mainly entailed in two of the mathematical practices: representations and exemplifications. Together with her understanding of the subject matter, and influenced by her considerations on students' learning characteristics and preferences, Mary first drew upon her knowledge of various representations of the quadratic formula and other methods: song, storyline, color coding, and other visual effects, which she believed would be suitable and helpful to students with different learning preferences.

The examples and exercises Mary provided to her class covered three basic types of quadratic equations, which could prove that she had a quite complete picture of the entire landscape of solving quadratic equations in the algebra curriculum.

Knowledge of learners' conceptions. Compared to her first two domains of MKT that were just discussed, Mary's knowledge of learners' conceptions was not as directly visible through the lens of the mathematical practices. Instead, her knowledge and considerations of her students' prior knowledge, skills, and learning preferences were identified mainly through the interviews, and maybe in combination with a few instructional activities in the lessons:

1. At the beginning of each lesson she always provided a short review of concepts or skills covered in previous sections, as a preparation for studying in the current lesson. She was fully aware of the fact that solving quadratic equations is a complicated task that requires students' proficiency in arithmetic and linear expressions and equations.

2. In Lesson 1 she asked the students to read the quadratic formula aloud, sing it as a rhythm, and tell the story of *The Sad Little Bee*, because she believed the purely symbolic derivation of the formula could be overwhelming or challenging to many students, whereas those verbal and rhythmic representations and activities would make it a lot more joyful and easier for the students to remember the formula. Students should like this because "it uses different memory devices".

3. With multiple examples and a series of exercises, she spent quite some time on demonstrating how to use the quadratic formula and what to pay attention to in carrying out the arithmetic computations step-by-step, which indicated that she was very knowledgeable of the various mistakes her students may make and the typical difficulties they would encounter in the calculations.

The Dynamics of MKT

Among the three domains of MKT demonstrated in Mary's teaching, her knowledge of the subject matter (i.e., the quadratic formula and other methods for solving quadratic equations) appeared to be solid and comprehensive. Together with her strong beliefs on the importance of facilitating students' conceptual understanding, her knowledge of those algebraic routines exerted most influences on her design and enactment of the three lessons. Since she truly understood the entire derivation process of the quadratic formula and its step-by-step justifications, and recognized the importance of the derivations and justifications, she was willing to spend much time helping her students to become familiar with these aspects of the formula. Similarly, based on her own clear insights in the features and applicabilities of the four methods for solving quadratic equations, and her preference on efficiency, she was both willing and feeling comfortable to lead the whole-class discussion on these matters.

Meanwhile, her lesson planning and teaching were also influenced by her knowledge of and past experiences with her students' learning characteristics and preferences. The constant review of previously studied foundational concepts and methods, introducing the quadratic formula with multiple representations, and detailed demonstrations of the execution of the quadratic formula on various examples, are all evidence of such knowledge and its influence.

Further, her knowledge of student learning in turn activated her third domain of MKT, knowledge of pedagogical representations - various ways to present and familiarize with the formula, and basic types of examples and exercises that could capture the nature of the formula and demonstrate the various applications of the formula.

Discussion and Implications

The three lessons taught by Mary provide an example of a mathematics teacher's attempt in teaching algebraic routines in ways that address both the conceptual and the procedural aspects of the routines. The MKT utilized by Mary shaped her design and enactment of the various mathematical practices which in turn created a positive learning environment that had potentials in facilitating conceptual understanding, procedural fluency, and strategic competence. Since only three lessons were observed and analyzed, it is unclear whether and when these aspects of mathematical proficiency could all be achieved simultaneously and effectively by the students. At the end of the third lesson, it was unknown to what extent the students in Mary's class had retained the conceptual underpinnings of the formula (derivation and justifications) and the features of all four methods for solving quadratic equations. It was obvious, though, that the majority of the students still needed more time to achieve fluency in applying the quadratic formula and to develop the kind of strategic flexibility that Mary expected. Simultaneously developing students' conceptual understanding, procedural fluency, and strategic competence in algebraic routines could be a fairly long and complex process. A teacher's solid MKT lays the foundation for effective teaching and learning, but may not be easily transformed into productive learning and expected outcomes in proficiency. Some major variants are involved during this transformation process.

One of the variants is the relative strengths of the three domains of a teacher's MKT. How these three domains of knowledge interact among themselves and how they collectively influence a teacher's thinking and decision making are worth further investigations. For instance, Mary's knowledge of the subject matter weighed more in her teaching than the other two domains of knowledge. Had she not understood the routines as in depth but instead had richer knowledge in pedagogical representations and student learning with respect to the routines, would she have come up with a different set of goals, focuses, and strategies and subsequently taught the lessons differently? And if the answer is yes, then how?

Another variant is a teacher's beliefs on mathematical proficiency, such as the nature and criteria for mathematical proficiency, the relationship among various aspects of proficiency, and how proficiency could or should be developed. How is a teacher's MKT of algebraic routines related to and interacting with her or his beliefs on proficiency? In Mary's case, her solid knowledge of the algebraic routines went hand-in-hand with her beliefs in pursuing conceptual understanding, which led to her attempt in developing students' procedural fluency and strategic competency based upon their conceptual understanding of the routines. Had she valued procedural automaticity more than anything else, several situations could have happened: she might have completely skipped the derivation, justifications, and comparisons, and focused more on the representations or applications; she might have simply let her students memorize the formula and then go through repeated practices until reaching procedural fluency; or, she might have told her students that all methods are good so they can choose whatever method they like to solve a given quadratic equation.

A more fundamental question is how to distinguish between a teacher's MKT and beliefs, or, when certain beliefs a teacher hold could be considered as part of her or his MKT. Not all teacher beliefs merit the label of knowledge (Feiman-Nemser & Floden, 1986). Following Plato's definition of knowledge as justified true beliefs (*Theaetetus*), Li (2007) identified five main sources of justification for knowledge claims: mathematical criteria, educational policies, educational theories or perspectives, collective experiences, and individual experiences and contexts. In Mary's case, her beliefs on building procedural fluency upon conceptual understanding and strategic competency could be well justified by the standard-based curriculum movement and state mathematics curriculum framework. In contrast, some of her notions about student learning characteristics and ways of presenting the algebraic routines appeared to be justifiable mostly by her personal teaching experiences. Whether these conceptions and strategies can become substantial knowledge claims that are easily shared by other fellow teachers and yield wider impacts on student learning, would largely depend on how well they match the policy, theory, and local educational settings that the other teachers are in.

Overall, there are multiple indicators of effective mathematics teaching (An & Wu, 2007) which may not always be attainable simultaneously. In making instructional designs and decisions to facilitate student learning, a mathematics teacher explicitly or implicitly puts varying preferences on her or his three domains of MKT, and assigns different interpretations and weights on the various mathematical strands of mathematical proficiency. Further, due to limited time and resources, the teacher may also have to constantly find compromises among personal MKT, the instructional demands from developing proficiency, and other variants such as policy and assessment requirements, district and school settings, diverse student characteristics and needs, and her or his personal beliefs, goals, expectations, and experiences.

A major implication of the study is that mathematics teacher preparation and professional development activities should address two crucial themes regarding teaching mathematical routines for proficiency. On the one hand, more emphasis needs to be put on the features and applicability of the various algorithms and strategies for carrying out a fundamental procedure in school mathematics: why, when, and how these algorithms and strategies work, in what cases they would work the best, when they would not work well, and how to compare them and choose the optimal one both in general and in a specific problem context. Scholars have summarized these features as *transparency*, *efficiency*, *generality*, *precision*, and *simplicity* (Kilpatrick, Swafford, & Findell, 2001) and provided suggestions for instructions. On the other hand, since teachers would have varying strengths and preferences across the three basic domains of mathematical knowledge for teaching routines, and teachers' knowledge and beliefs with varying sources and degrees of justification would exert different impacts on teaching, mathematics teacher preparation and professional development should not only aim at strengthening teachers' MKT in all three domains, but also help teachers to develop the insights, skills, and flexibilities that are needed to make well-informed, thoroughly-reasoned, and balanced decisions in promoting mathematical proficiency among all mathematics learners.

The kinds of MKT revealed this study may better to be phrased as mathematical knowledge teachers draw upon *in* teaching, as they are observed empirically only from individual teachers' work on a particular mathematics topic with students in individual classrooms. The analysis and synthesis of these kinds of knowledge were certainly not meant to be exhaustive. Future studies should involve other teachers with distinct backgrounds, experiences, and classroom settings and teaching other types of mathematical routines. Only in doing so may we be able to ultimately generate and describe a more comprehensive body of mathematical knowledge that the majority of mathematics educators would collectively agree as essential *for* effectively teaching mathematical routines for proficiency.

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