Journal of Mathematics Education
June 2013, Vol. 6, No. 1, pp. 17-37

Teacher Change in Re-Evaluating Their Understanding of A Numeracy Cognitive Framework in New Zealand

Peter G. Hughes
Kumar Laxman
University of Auckland, New Zealand

The presence in the middle school of a number of New Zealand students who need remedial help in mathematics, particularly in the area of number sense, suggests it is desirable to relook at the ways in which teachers have traditionally taught mathematics content knowledge in class. An examination of research into how teacher change can be facilitated to enhance the quality of instructional delivery was used to inform our study which encouraged teachers to reflectively better understand and apply the cognitive framework in the teaching of numeracy. Teachers' awareness of how their year four students' mathematical thinking, particularly in the area of number, increased through interviewing students in their own classes is described in this paper. This helped participant teachers to see hitherto unperceived complexities in students' construction of mathematics concepts and enabled the teachers to subsequently experiment in their classrooms the teaching of numeracy in more meaningful ways.

Key words: Numeracy, teacher reflections, mathematical thinking, cognitive framework

A number of studies by Young-Loveridge (1987, 1989) raised concerns about the stage of achievement in mathematics, particularly in the area of number, of New Zealand school students aged in year 1 to year 5 classes. Further the *Third International Mathematics and Science Study* (Garden, 1997) reported that internationally New Zealand was not performing well in mathematics. A consequence was that a teacher advisory group appointed by the Minister of Education recommended back to that professional development in teaching maths was advisable for teachers of year three students. Consequently the Ministry asked the six traditional teacher-training institutions around New Zealand to tender for contracts in their region for professional development for teaching mathematics for teachers of year three. In the Auckland region the proposal was based on teachers university study in a paper that included material from a professional development programme for teachers in New South Wales called Count Me in Too

(NSWDET, 1999). In turn this was based on a mathematics intervention programme (Wright et al., 1993). Conducted in New South Wales, and known as Mathematics Recovery, the programme was modelled on the Reading Recovery programme (Clay, 1987). The theoretical basis underlying this mathematics programme is a six-stage model of early arithmetical development adapted by Wright (1989, 1991) from research conducted by Steffe and Cobb (1983) and Steffe et al. (1988). This model is shown in Figure 1.

Stage	Behavioural Indicator
0 Preperceptual	When attempting to count is unable to co-ordinate number words with items
1 Perceptual	Can count visible collections
2 Figurative	Can solve additive tasks involving screened collections but counts from one when doing so
3 Initial Number Sequence -Sequential Integrations	Counts-on to solve additive and missing addend involving screened collections
4 Implicitly Nested Number Sequence - Progressive Integrations	Uses counting-down-to solve subtractive tasks and can choose the more appropriate of counting-down-to and counting-down-from
5 Explicitly Nested Number sequence - Part/whole Operations	Uses a range of strategies which include procedures other than counting-by-ones such as compensation, using addition to work out subtraction, and using known fact such as doubles and sums which equal ten

Figure 1. *Arithmetical stages.*

This model represents a marked change from traditional models of teaching in that teachers, instead of observing outcomes that are basically right or wrong, are expected to understand there is a hierarchy of cognitive skills that Steffe's research undoubtedly demonstrate is present, and they need to take this into account in their teaching. For example, to solve 9 + 4 Stage 2 Figurative students would be able to count from 1 to 13, Stage 3 Initial Number Sequence students would count on: 10, 11, 12, 13. However, Stage 5 Explicitly Nested Number sequence-Part/whole Operations students would say something like 10 + 4 is 14, and 14 minus 1 is 13. This paper reports on how readily teachers would understand then use this hierarchy.

Because Count Me in Too was a years one to three project in New South Wales, and the New Zealand Numeracy Projects would eventually cover from years one to ten, extra stages were added, and the opportunity was also taken to simplify the names of the stages and their behavioural indicators. Figure 2 shows this modified framework (MoE, 2004).

Stage	Characteristic of Stage
0 Emergent	Students at this stage are unable to consistently count a given number of objects because they lack knowledge of counting sequences and/or the ability to match things in one-to-one correspondence.
1 One-to-one Counting	This stage is characterised by students who can count and form a set of objects up to ten but cannot solve simple problems that involve joining and separating sets
2 Counting from One on Materials	Given a joining or separating of sets problem, students at this stage rely on counting physical materials, like their fingers. They count all the objects in both sets to find an answer.
3 Counting from One by Imaging	This stage is also characterised by students counting all of the objects in simple joining and separating problems. Students at this stage are able to image visual patterns of the objects in their mind and count them.
4 Advanced Counting	Students at this stage understand that the end number in a counting sequence measures the whole set and can relate the addition or subtraction of objects to the forward and backward number sequences by ones, tens, etc.
5 Early Additive Part-Whole	At this stage, students have begun to recognise that numbers are abstract units that can be treated simultaneously as wholes or can be partitioned and recombined.
6 Advanced Additive Part-Whole	Students at this stage are learning to choose appropriately from a repertoire of part-whole strategies to solve and estimate the answers to addition and subtraction problems.
7 Advanced Multiplicative Part-Whole	Students at this stage are learning to choose appropriately from a range of part-whole strategies to solve and estimate the answers to problems involving multiplication and division.
8 Advanced Proportional Part-Whole	Students at this stage are learning to select from a repertoire of part-whole strategies to solve and estimate the answers to problems involving fractions, proportions, and ratios.

Figure 2. *Extended arithmetic stages.*

This rather new notion that cognitive frameworks can help teachers better understand their students' strategic thinking in number became the central focus in the professional development in the New Zealand Numeracy Projects. Simultaneously with the professional development of teachers in understanding and using the number cognitive framework in their teaching, teachers were also exposed to constructivist ideas of teaching. Our study elucidated in this paper looks at the pedagogical impact of Steffe and Cobb's framework in enhancing the skills and quality of the teaching of numeracy in New Zealand classrooms.

Constructivist Pedagogy

At the core of student-centred mathematics teaching is a constructivist epistemology. Means and Knapp (1991) challenge the effectiveness of conventional approaches in mathematics teaching. They see conventional instruction as helping students master low stage basic computation before moving on to work requiring the higher stage thinking involved in reasoning and problem solving. While they observe there is evidence that conventional teaching leads to gains in low stage thinking they report very little gain in high stage thinking. They note that current cognitive psychology suggests a new attitude towards learners of all abilities is needed. Rather than a teacher offering explanation and guidance using small, carefully sequenced steps they suggest instructional strategies should be based on solving more complex problems; the teacher models powerful thinking strategies, encourages multiple approaches, provides scaffolding to enable students to accomplish these more complex tasks, and make dialogue central to teaching and learning.

Well-planned traditional teaching is typically top-down, clear, and well organised; explanations are full, logical and easy for the student to follow. Everything is provided, and there lies the problem for constructivists; they argue that the results of such a programme are inevitably shallow. Explanations by the teacher will not transform novices into experts. The learner has not undergone the struggle to find meaning that is a necessary condition of learning. A consequence of accepting these constructivist arguments about the nature of learning is that different principles than are traditionally used are required in teaching mathematics.

The following set of teaching principles is an attempt to help teachers turn general constructivist principles into useable ideas for the New Zealand Numeracy Projects:

- The teacher will find the highest stage of students' current thinking in terms of the cognitive framework and teach to this stage of thinking, rather than by analysing behavioural outcomes. This is the *core* principle in this study.
- The teacher will present students problems that are within their zone of proximal development, that is to say problems are selected from the cognitive stage that students are can reliably operate at. For example, students who are operating at stage 5 (Figure 1 or Figure 2) can easily learn to solve problems like 48 + 6 by adding and subtracting 2 to give 50 + 4, which is 54 because this problem is within the their zone of proximal development. But a problem like 88 + 88, solved mentally, would be regarded as a Stage 6 problem (Figure 2) that would consequently be outside the zone of proximal development for stage 5 students.

- The teacher will set problems or tasks for students to solve in groups - the teacher will not guide the students towards solving problems by pre-set methods.
- The teacher will tell the students things that are information, but will *not* give clues on how to solve problems. For example, if year four students are asked to solve thirty-two plus eighteen presented orally some students typically need to be told that eighteen means ten plus eight plus teen; however the teacher does not give hints on a solution method
- The teacher will not ask questions during group discussions, rather the teacher will listen in to detect possible problems.
- When gathering groups of students together as a whole for discussion of the problem the teacher will not act as a validator of answers; rather she or he will use active listening whereby she or he repeats student solutions with greater clarity filling in the implicit gaps in the logic the used, but she does not present a different solution method. Active listening would normally be expected to provoke the realisation by students that something is wrong in their reasoning without being told this by their teacher

$$\begin{array}{r} 53 \\ -\ 28 \\ \hline 35 \\ \hline \end{array}$$

Figure 3. *Common subtraction error.*

An example will help to illustrate why such constructivist teaching is desirable. Figure 3 shows a mistake in subtraction frequently made by year four students. In traditional teaching the teacher would point out the error to the child, namely that the child has taken the smaller number from the bigger number in each column. Then the teacher will demonstrate a correct written method of subtraction. Finally the child will practice similar problems until it mastery has been achieved. Unfortunately this approach encourages shallow schema formation. Equally seriously it often fails to find the root cause of the failure. In this case this diagnostic question often detects what the real problem is: students are asked to count out fifty dollars in ten dollar notes. Having done this successfully the students are then asked to say how many ten dollar notes they have. Many of them, who count from one, fail to understand fifty means five tens. Such students can conceive of 53 only as 53 singletons and not 5 tens and 3 ones. Hence even if a traditional teacher gives an explanation with concrete apparatus that the key step in the decomposition method of subtraction involves renaming 53 as 4 tens and 13 ones it is unlikely to make any sense to these students. Thus an important principle in the constructivist teaching is that teachers needs to begin their teaching by knowing the conceptual difficulties students have rather than treating superficial symptoms.

To continue with the example above: suppose the diagnosis has been made that the major problem is that students have is a poor understanding of the place-value system. Constructivist thinking suggests a very different prescription for teaching from the tradition one. Traditional teaching is likely to involve showing the child a table headed with tens and ones and asking the child to show two digit numbers with some kind of structured material. A major constructivist criticism of this method is that the child is being guided and controlled by the teacher; in these circumstances the struggles, wrong turnings but eventual construction of a robust schema, that constructivists believe is the sine qua non for real learning, have not occurred.

How then might the teacher who has taken to heart the criticisms of the tradition methods proceed? Vygotsky's (1978) notion of the Zone of Proximal Development (ZPD) will prove helpful. By the ZPD Vygotsky means the distance between what the child now knows and what she could know with the aid of an expert. A consequence of combining the notion the ZPD with believing that the child must undergo significant cognitive conflict to deconstruct poor schemata and construct better ones, is that the teacher, having determined a student’s ZPD, must pose challenging complex problems for the student that are selected from within the child's ZPD. In our example students who cannot subtract well because of poor place-value knowledge needs help to construct a reliable number schema. Having inferred that place-value is within the ZPD of the child the teacher would pose multi-digit (not just two digit problems) involving addition and subtraction; these problems could be solved on some concrete material such as play money and the answers checked by the child with a calculator. The teacher would not suggest how to solve the problems. Further, when the child is obviously thinking the teacher will not ask "tell me what you are doing?" Such problems will offer a significant challenge to the students; they must model the numbers on money correctly, deal with the exchanges that require ten notes to be swapped with one note of the next higher denomination (or vice-versa), recode what the money shows as the answer, and check this number with a calculator. The potential for making errors is very high, but this is an advantage not a disadvantage. Provided the problems are within the students' ZPD they will solve the problem; in so doing they will accommodate new knowledge, that is to say they will construct a new larger and more flexible number schema.

Aims of Study – Evaluating Teacher Beliefs and Changing Practices

It is likely, perhaps even probable, that constructivist assumptions about the nature of learning that are at the core of the New Zealand Numeracy Projects are significantly at variance with the beliefs many teachers have about teaching. Hence an essential part of any teacher training programme is to identify and address the conflict between teachers' current beliefs and constructivist pedagogy. The study examines this aspect of teacher change by encouraging participant teachers to reflect upon their current beliefs so that a critical self-evaluation of the efficacy of their mathematics teaching practices can be accomplished.

Extensive research that has focused on the process of change in teacher beliefs indicates that a simple model in which the teacher is exposed to some in-service training and then implements change in her classroom is seriously flawed. Guskey (1985) suggested that, in curriculum innovation, the organisers of in-service programmes frequently use the following model to explain how student learning will improve (Figure 4):

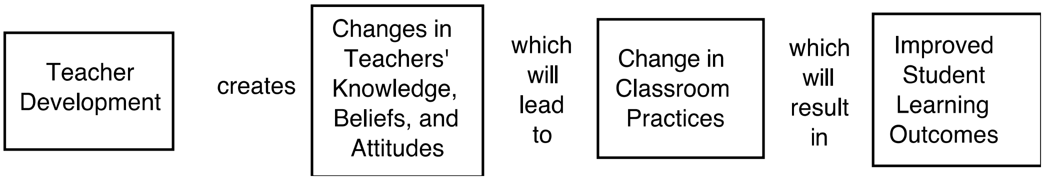

Figure 4. *Traditional PD model.*

However Guskey proposed that this model does not really represent the real process of teachers changing their teaching methods. Noting that what determines the actual practices used by an experienced teacher is her or his experiences in the classroom Guskey suggests that change in classroom practice drives change in teachers' knowledge, beliefs and attitudes. Thus he proposed a reordering of the model above (Figure 5):

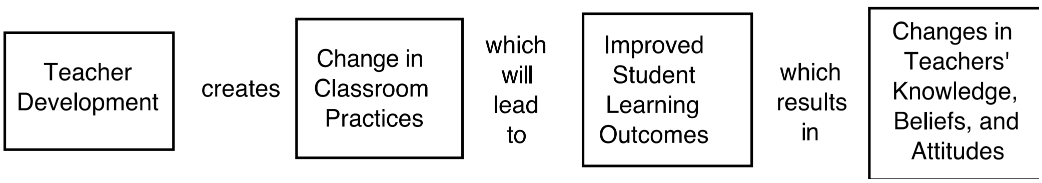

Figure 5. Classroom based PD model.

Subsequently Clarke and Peter (1993), expanding a model developed by Clarke (1988), proposed that the process that drives change in teacher knowledge belief and attitude is a result of more complex interactions that the simple linear model proposed by Guskey (1985); no one model will work for all teachers. However they agree with Guskey that teacher experimentation is *one* possible starting point that outside agencies such as ourselves can act as stimulant to improving teaching and therefore learning. This study uses this model.

Accepting that teacher change may be driven by teacher experimentation in their classroom, it is not obvious what will cause her to experiment, if she is not already doing so. In this study it is posited that interviewing students from the teacher's own class about their mathematical thinking using the cognitive stages model (Figure 1 and 2), then teaching these students in areas where problems are revealed, may be a stimulus for the teacher to experiment with her teaching, if the interviews and teaching sessions reveal that the mathematical thinking of her students is significantly different from that which she had previously supposed to be the case.

Constructivists (Tanase, 2006) suggest that, to move towards a constructivist teaching practice, teachers need to engage in doing mathematics themselves; they should experience constructivism as a learner before they can understand constructivism as a teacher. Our study investigated how teachers, who felt otherwise confident about their own mathematical ability, recognised a need for an alternative approach to their current teaching practices and experimented with new approaches that subsequently had significant effects on changing their beliefs about teaching and learning.

Methods

Research Questions

This study sought to

- Investigate impact of reflective observations by teachers of their own students in attempting to understand Wright's cognitive framework (Figure 1 and 2) in the New Zealand Numeracy Projects
- Examine the effects of applying the cognitive framework in teachers' classroom practices to improve their pedagogical delivery of Mathematics curriculum

Research Design

A case study approach was adopted in this study to examine the specified research questions. A case study by definition constitutes an exploration of a 'bounded system' or a case (or multiple cases) over time through detailed, in-depth data collection involving multiple sources of information rich in context (Creswell, 1998). A case study method was used since it enabled the researchers to focus on the central research phenomena of teachers' thinking and teaching approaches towards the learning of numbers. A particular case serves the real purpose and objectives of discovering, probing deeply, gaining a rigorous insight and understanding of a chosen phenomenon (Burns, 2000). The case to be selected would generally be used to optimize the learning of complex meanings and interpretations encapsulated in the case within the amount of time that was allocated to the study (Bogdan & Biklen, 1998).

Subjects

The two teachers in this study were both at the same primary school. Ms A undertook teacher training directly from high school. This was her second year of teaching. Ms B also undertook teacher training directly from school, and was in her fifth year of teaching; she became a Senior Teacher after three years teaching.

Students in the study were all from year four with two exceptions; these two students were in year three. Each teacher picked from his or her respective classes an able student, an average student, and a lower ability student for interviewing; the teachers used their own judgment about what constituted able, average and lower ability. All the students were from year four except one who was in year three; his teacher selected him because she regarded him as better at maths than any of her year four students. Permission was obtained for the students to be part of the study from their parents.

Procedure and Materials

At the start of the project the two teachers were interviewed separately and these interviews were audio recorded and later transcribed. They were asked to outline their current classroom practices in teaching mathematics, and how they determined which students have limited understanding in the area of number. They were each given a log book to record their observations and reflections after each session in the study. As part of the preparation for the interviews, the teachers met twice to review transcripts of a range of case studies based on a similar survey of year four students' number knowledge.

The three students selected by each teacher were given a number survey consistent with the assessment teacher used the New Zealand Numeracy Projects (MoE, 2008). Its main purpose was to detect the strategic stages that the students were operating at i.e. to determine their ZPDs. Each student was then interviewed individually by her teacher and the researcher. Special care was taken to ask the students how they obtained the answers, and not to guide them towards, or teach them how to get, correct answers.

Subsequent to these interviews each teacher selected three more students from their own class whom they identified as being below average in their knowledge of number. The process of interviewing each student about her/his methods of solution was repeated; the researcher was not present for these interviews.

During the period in which the student interviews involving each teacher were taking place the teachers, along with four other teachers not reported on here, attended two mathematics content sessions run by one of the researchers. The researcher taught the participating teachers some unfamiliar mathematics content using the constructivist teaching model. The teachers were asked to imagine that they were in a school for octopi and hence had to do their arithmetic in base eight. They were then told they were going to learn, from first principles, much of the mathematics content of the interview their students had done except that all the work would be in base eight. Calculators that do octal (base eight) arithmetic were provided. The teachers always worked in a group; they were encouraged to discuss methods of solution and come up with mutually agreed answers. Checking answers using the base-eight calculator was encouraged. The researcher did not validate answers; validation was effected through group interactions. All problems were to be presented without any accompanying hints as to methods of solution.

By the end of the two base-eight sessions each teacher had interviewed a total of six students from her class. As a result of these interviews each teacher decided which, and how many, students would participate in the next stage of the study, namely fixing problems in number uncovered in the interviews.

Finally, at the end of the study, each teacher was asked to re-read the transcript of her initial interview, to look through her log book, and then reflect on any changes or additions she had made in her practice of teaching of mathematics. They were then re-interviewed and the interviews were taped and transcribed.

At the heart of the qualitative case study data analysis process was identification of key themes. Themes that shed light upon the research phenomena of numeracy learning being investigated were analyzed for their key meanings and applications. The pedagogical setting, contexts and participant perspectives of conceptual thinking and processes embodied in the learning of numeracy were scrutinized for meaningful interpretations to be drawn. Underlying patterns in thoughts, behaviours and experiences related to the research topics being probed were studied in order to compare and contrast thematic categories providing insights and answers to the specified research questions.

Results

Case Study One - Ms A

As a result of discussing the transcripts of year four students from another school, and interviewing six students from her class about their number thinking, Ms A believed that learning mathematics was more complex than she had previously thought:

> There is more to the teaching and learning of mathematics than meets the eye. There were obvious differences in the way that able and less able students solved maths problems. Help! Especially with Jean. *(The interview with Jean showed that she did all single digit addition problems by counting on with her fingers. She got no other questions in the survey correct.)* Also the interview showed gaps where I had not thought there would have been any. For example Jim. *(The interview with Jim showed that he had correct answers to almost all questions. Nevertheless Ms A was surprised to find that Jim did not use stage five strategies to work out single digit addition problems even though he could instantly recall combinations of ten and the doubles.)*

Before the interviews Ms A believed that the strategy stage that students operated at for single digit addition facts was not important:

> It is going to be faster (to use stage five strategies) but it doesn't matter. The way I think if they are using the fingers now, eventually they are not going to, there is no point in me saying you can't use your fingers.

However, as a result of seeing from the first three interviews an unexpectedly wide range of strategies used by students in her class, Ms A decided she would teach these strategies, with some success for average and able pupils:

> From (my teacher training) I had always seen the usefulness of teaching strategies, but within the classroom I haven't really done it until this year. So it's not that I haven't believed in it, it is just that I haven't believed it enough to actually do it. But now I've tried it, it has helped the majority, especially the more-able students; it has helped them get their thoughts together faster and put some logic behind it. With the middle students, yes it has helped them. The weak ones, they don't see the logic that

> you have got eight plus five - you take two from the five to add on to the eight, they just can't.

Ms A was confident about her own stage of mathematical knowledge and ability; while finding the sessions on base eight arithmetic easy they nevertheless highlighted for her difficulties some students have in learning number ideas:

> I didn't find that exercise itself hard, but because of the challenge of the same rules (as base ten) but in a different way, it made me aware it's not as easy as it looks. We are so used to base ten, so what we do seems really logical and if it is that logical why aren't (the pupils) picking it up quickly?

Following the six interviews Ms A identified Kesia as a person in need of special help; Ms A and the researcher agreed they would try to teach some misunderstood aspects of number. Kesia's survey and interview with Ms A indicated she indeed did have some significant difficulties. She could instantly recall the doubles facts but made no attempt to use them to derive other facts; her main strategy to add single digits was to count on from the larger of the two numbers. Kesia could correctly identify pictures showing money up to $99 but was confused by three digit numbers or higher; $320 was written as 3020 even though Kesia said 3020 means three hundred and twenty. Kesia showed significant lack of number sense when she apparently failed to detect that answers to two digit additions and subtractions could not be correct. For example:

$$\begin{array}{r} 51 \\ -\ 17 \\ \hline 46 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 27 \\ +\ 8 \\ \hline 115 \\ \hline \end{array}$$

Figure 6. *Kesia's errors.*

For Kesia lack of understanding place-value was a major issue. In the teaching sessions Ms A observed that progress was made unevenly. For example, when asked to add 23 and 89 with concrete materials Kesia recorded the answer as 10 hundreds, 2 tens and 2 ones even after previously solving similar problems correctly. Examples like this over the year of the study helped Ms A construct a more complex view about the nature of mathematics learning:

> From doing this (study) part of my thinking has changed. I used to assume that maths learning is sequential even though they might have to go back a few steps to re-do it.

> I don't think that any more. I don't see it as linear. There are lots of little points that make up the whole but you can't put it in a line and say well if you learn this one, then this one, that you'll eventually get to the top. I can't think of a model at the moment that would help somebody teach that particular thing - which makes it really hard.

Subsequent to teaching students from her class individually Ms A wanted to find ways to experiment with teaching more than one child. The researcher and Ms A agreed to try teaching four students as a group. Ms A selected four students to work on multi-digit addition and subtraction. The school provided release time for Ms A and the students were withdrawn from class. Ms B indicated she was keen to attend and observe; this was agreed to.

Although the researcher preferred setting problems he believed were within the Zone of Proximal Development of the four students, and simply letting them solve problems through their own efforts with little or no teacher intervention, it was agreed by negotiation with Ms A, to use a much more structured model. Initially a three-digit addition problem was set to be solved with play money and each child was assigned a role. There was a banker, a recorder of the answer, a person who paid the money, and a person who received the money. With each new problem the roles were rotated. Ms A was not happy with the degree of students' involvement so the researcher suggested that the students attempt some multi-digit subtraction problems without any roles being assigned. This suggestion was consistent with constructivist teaching; the students themselves decided how to proceed with the problems and to validate the answers themselves. The stage of pupil involvement now markedly increased; after the session Ms A now thought no assignment of roles to be superior to assigning roles:

> (When they had roles) the students didn't give other students any feedback. And if they got stuck they were left to try and sort it out with that on their own a bit more. Whereas when they didn't have their roles they were able to "Oh, well this is where you've forgotten to do this, or how about, you've got ten of these, what do you need to do?" (The unstructured way) definitely changed the dynamics; they were all into it when they didn't have their role. I think maybe with the roles they were waiting for their turn.

As a consequence of these group sessions Ms A indicated that she would like to try teaching mathematics to her class using groups. The researcher gave her a copy of Marilyn Burns (1990) "The Math Solution: Using Groups of Four" as a source of possible ideas for group teaching.

Burns suggests that classes be organised into co-operative learning groups of four. The teacher provides the whole class with problems that the groups then try to solve. The solutions are summarised and the groups share with the class their processes, procedures and strategies and their solutions.

Ms A created nine groups of four in her classroom. When teaching operations with numbers she set word problems using numbers that were much larger than usually given to year four students. For example a problem like this was given: "Thomas has $7455 and his sister has $7779. How much do they have altogether?" The fact that this problem needs multiple exchanges on play money was deliberate; this was designed to promote significant discussion and interaction within each group. Validation of answers was by calculator. How the answer was recorded was left to each group to decide. For example, for the addition problem above, two methods of recording the solution were:

ten th	th	hun	tens	ones
1	5	2	3	4

and

$$\begin{array}{r} 10000 \\ 5000 \\ 200 \\ 30 \\ 4 \\ \hline 15234 \\ \hline \end{array}$$

Figure 7. *Two methods students used for place value recording.*

Case Study Two - Ms B

As a consequence of the interviews of students in her class Ms B altered her opinion of the importance of students having stage five strategies for single digit addition.

> (At the start of the year) I really didn't use any (strategies). For me I automatically knew that nine and seven were sixteen, and I was probably trying to teach it in a rote method. Like we do counting and clapping games, "Nine and nine is eighteen," etc. Whereas now we are looking at strategies like "using the doubles". It gives (the students) a far greater repertoire rather than relying on instant recall through rote, and they can apply it to larger equations. They are incredibly proud of explaining how they did something as well.

During the base eight arithmetic sessions Ms B improved her personal knowledge of place-value concepts with consequences for her classroom teaching of mathematics.

> I've always used the numbers in my life and not actually understood what the number meant. I understand (place-value) better now. It was brought home to me because I was doing (place-value) at the time in class. I've become a lot more patient with the kids. I have, more understanding as to why they can't understand something which I initially found very basic, and why couldn't they understand it.

At the end of the second base eight session Ms B requested that the researcher teach some of the students in her class. This was agreed to; Ms B selected two students from her class for this purpose. Sally was one of the students selected.

$$\begin{array}{r} 53 \\ -\ 19 \\ \hline 46 \\ \hline \end{array}$$

Figure 8. *Student error*.

From the interview it was evident that Sally had a good understanding of place-value but had some difficulties with the subtracting with multi-digit numbers. She was withdrawn from class for two sessions. Release time for Ms B was provided by the school. In previous sessions Sally showed that she could accurately read numbers into the thousands, and she was able to show these numbers with play money without hesitation. However, when Sally was asked to complete 63 - 19 in the vertical algorithmic form she could not do it correctly despite having been taught how to do two digit minus two digit problems by Ms B in the previous week. Her attempt is shown in Figure 4. Suspecting that Sally did not understand the renaming process used in the subtraction algorithm the researcher wrote 53 - 19 = on a card and asked Sally to solve this problem. Her attempt to solution confirmed the researcher's suspicion: Sally had $53 and had to pay Ms B $19 using one and ten dollar play money. Sally correctly counted out 6 tens and 3 ones from the bank. She gave Ms B a ten and 3 one dollar notes. Sally realised that she had to give Ms B another 6 one dollar notes. She solved this problem by taking 6 ones from the bank. After the researcher asked her to give back these 6 dollars as they were not hers she presented a ten dollar note to the bank for exchange; but she only asked for six dollars for her ten dollar note. When this confusion had been resolved Sally was presented with a series of two and three digit subtraction problems that she solved using the money and she validated the answers with a calculator. The researcher did not offer hints to help solve the problems. When, as sometimes happened, her answer was in disagreement with the calculator she resolved the conflict for herself. In the second session Sally learned how to connect the actions on the play money with writing two digit subtraction problems in the standard vertical from.

Subsequent to the teaching sessions with Sally Ms B had altered some of her opinions about mathematics teaching:

> I now try to make activities much more real-life and present a problem for them to solve. So it is far more child-centred than teacher-centred with me providing a lot of algorithms. I saw (the researcher) modelling what you were doing, where the real-life, when I saw that it worked, and the child just clicked so much more quickly, as to what was happening. And it just seemed practical and obvious that was what we had to do. And I don't know why I wasn't doing that before. For example what I used to do on addition and subtraction they might have ten problems (given in the vertical form) involving re-naming. Then at the bottom I might put one or two applications. Whereas now I present the whole thing in problem solving form. Because everything needs to be associated with the problem. Before the two were very separate and the students weren't relating the two to each other.

Ms B left on an extended overseas trip soon after the teaching sessions, so she was able to implement only some planned changes in her teaching. However she did state her intentions for teaching maths in the next school year:

> I think that perhaps next year, I can pull out small groups of students and focus on them, because it is impossible to do money with a group of twelve kids who might be at that stage in maths. I would allow for the peer teaching/tutoring, the peer evaluation with learning centre activities and pull out that small group because it is so much more effective. [Previously] I was working with three or four groups; some groups did have a dozen students in them and I was trying to do the entire thing with all of them. They were all at the same stage; they all were having trouble with the decomposition in subtractions. Some were working with the money and they all had their money in front of them. But to actually physically see what was happening and that each child was counting up correctly and each was at the same stage of getting the money, and just physically manoeuvring the equipment, it wasn't as effective as I've been working with groups of two or three kids.

Ms B came to believe students could achieve more in mathematics than she had previously thought:

> I think I always had that fear that [year four] students were at a dependent stage. And not being overly familiar with the kids at the year 1 and year 2 levels I probably didn't give them credit that they could be taught to be independent.

In teaching sessions Ms B valued not confirming or correcting students' answers:

> I liked the way that (the researcher) didn't affirm right or wrong answers, but (the researcher) used the calculator as a check. And again that's a huge motivation, and often I've seen through work that the kids do trust the calculator's judgment and then they'll go back themselves - so it's very child-orientated. "Oh, that's right, that's what I did incorrect!"

Ms B experimented in her classroom with allowing students to work through problems for themselves and validate answers independently of the teacher. Despite some initial misgivings Ms B was pleased with the results:

> A child who is scared of the teacher's reaction knows I'll collect the books at the end if I don't have a chance to see them during the lesson. They are very honest too. I debated at the start when they were marking their own work they would change an answer. Initially I just did not trust certain students, the ones who I knew were less confident and wouldn't want me to know that they'd got something incorrect. The ones who always needed to be right. The odd child of course does still, but the percentage is a lot lower.

Ms B was aware before the project began that many students have difficulty in moving from c materials to abstract written forms. Ms B was acutely aware that this was evident in the session where Sally trying to work out 63 – 19. Despite having been taught in class in the previous week Sally made serious errors:

> Teachers often move students too soon. They seem to be fluent with two digit with the money, two digit say subtraction and addition, so they move the students off immediately to abstract and they'll do it. And then

> perhaps the teacher moves on too. "Three digit, surely you can do three digit now that you have done the two digit," without going back to concrete material.

Discussion

As a result of interviewing students with a range of ability, Ms A became more aware of the range of thinking strategies used by students in her class; she began actively to teach a wider range of strategies. While personally not finding the content sessions very challenging, the experience of being in the position of the learner rather than the position of the teacher increased her awareness of, and sympathy for, students experiencing difficulty with their number sense. Following the intervention teaching sessions with students from her class she began experimenting with new whole class teaching techniques that were markedly more constructivist than her previous methods; in particular she adopted group teaching using significantly more challenging complex problems than she had previously used. She came to believe that learning mathematics was more complex and non-linear than she had previously thought.

After interviewing students with a range of ability Ms B decided to actively promote stage 5 strategic thinking for in her class; Ms B found the base eight content sessions very challenging; she personally improved her understanding of the notions in place-value within consequent improvement in her teaching of the topic. Following the intervention teaching sessions with students from her class she suggested that she would move towards markedly more constructivist teaching when she returned from her overseas trip. She now believed she had less to need to control the learning, rather allowing students more freedom to make their own mistakes and correct these mistakes themselves. A major conjecture in this study was that exposing teachers to what mathematical thinking the students in their class were using would propel the teachers into experimentation in teaching mathematics; the conjecture proved correct in the case of Ms B.

Conclusions

Research indicates that teacher change is driven by teacher experimentation; in this study it was posited that the stimulus to engage in experimentation would come from the teachers interviewing pupils from their own classes. This indeed proved to be the case. Both teachers reacted sharply to increased awareness of the range of strategic stage used by their own year four students; in both cases the teachers moved from assigning little importance to high stage thinking strategies to actively experimenting with ways of encouraging such high stage thinking in their classes.

For both teachers the personal content sessions on base-eight arithmetic had a significant impact. By being put in the position of learners within a constructivist framework both came to a realisation that learning mathematics was more complex than they had previously supposed; both commented that they had increased sympathy for the problems students have in learning even apparently simple mathematics. Improving their content knowledge had a significant impact on the teachers' willingness to experiment with new forms of teaching.

By observing that their students were operating differently from expectations both teachers moved towards significantly more constructivist teaching. The key feature was they both surrendered much of the control of the student learning to the students. They both became more task-driven where students discuss and work on problems extensively before the teacher brings the students together to examine the key ideas of the problem.

While it is not the prime purpose of this paper to discuss the effectiveness of the New Zealand Numeracy Projects a large number of Ministry of Education commissioned papers published between 2001 and 2008, tend to reinforce the conclusion that getting teachers to recognise the existence of a cognitive framework and its use has been valuable. (These are available by Googling the NZMaths website.) However it would be wrong to suggest that the papers indicate much uptake in the use of constructivist teaching methods raised in this paper, and this indicates the need to address this issue in any future projects.

As a postscript there were very significant and unexpected pieces of research relating the Numeracy Projects to algebra learning. (Irwin & Britt, 2005a, 2005b, 2006). What the researchers found was that pre-secondary students who mastered stages 6 and 7 of the framework, that is to say advanced mental methods of addition and subtraction and multiplicative thinking, went on to secondary school where they were found to perform significantly better in algebra than students who had not mastered these levels of thinking. And even more significantly this advantage continued even secondary schools had nothing to do with Numeracy Projects.

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Authors:

Peter G. Hughes
University of Auckland, New Zealand

Kumar Laxman
k.laxman@auckland.ac.nz
University of Auckland, New Zealand