

# Using Community College Students' Understanding of a Trigonometric Statement to Study Their Instructors' Practical Rationality in Teaching

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*The purpose of this study is to investigate how community college mathematics instructors reasoned about information regarding students' understanding of trigonometric ideas. We sought to create a dissonance between what instructors thought their students understood about trigonometry and what the students revealed through paper and pencil tests and in-depth interviews. This is part of a larger project that seeks to understand the conditions under which proposals for reforming instruction in tertiary mathematics are effective. We believe that this knowledge is foundational for the development of appropriate learning experiences for faculty teaching mathematics in community colleges, a very special tertiary education institution in the United States.*

**Key words:** trigonometry, community colleges, teacher professional obligations, students' understanding

Community colleges educate about 50% of all undergraduates in the U.S. (Dowd et al., 2006) and nearly 46% of all undergraduate *mathematics* students at U.S. colleges and universities (Blair, Kirkman, & Maxwell, 2013). Despite this, surprisingly little research exists on the quality of community college mathematics education. Historically, these institutions have assumed four functions: (1) academic transfer preparation, (2) terminal vocational certification, (3) general education leading to an associate's degree, and (4) community education. Collectively, the efforts' goal has been to accomplish three diverse aims: democratic equality, social mobility, and social efficiency, which for some critics, (e.g., Labaree, 1997) results in unresolved tensions over the central aims of education in the United States (p. 196). Current shifts in economic organization have even added a fifth function, that of re-training workers for a changing economy. These multiple and competing aims and

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functions set the community college apart from their counterparts in four-year colleges and universities.

Unlike four-year colleges and universities, community colleges operate with open access policies that create classrooms with full- and part-time students of all ages and backgrounds. For instance, a majority of these students are under prepared and under resourced and have family and work obligations. In addition, the populations at many community colleges also have high concentrations of students with disabilities and low English proficiency (Goldrick-Rab, 2007). Lastly, faculty in community college are not expected to conduct research but to concentrate on *teaching* (Grubb, 1999). This results in heavy teaching loads (e.g., 4 to 6 courses per term for full-time faculty, for an average of 15 credit hours) and more demands for doing administrative work (e.g., advising).

Even though community college faculty plays a large role within the higher education arena, their pedagogies and their thinking about teaching and learning has rarely been explored in the literature. Little is known about the decision making processes of community college mathematics faculty. Even less is known about the nature of their knowledge of their students' thinking and ways in which their instructional planning may be influenced by such knowledge. The purpose of the work we report here was to address this gap. In this particular study we seek to better understand the nature of pedagogical knowledge and decision making of the community college mathematics instructors. Specifically, we asked how do community college mathematics instructors responsible for teaching trigonometry interpret information about their students' understanding of inverse trigonometric functions? How do they use such information to plan their lessons?

### **Theoretical Background**

The study draws from the theory of practical rationality, a construct coined by Herbst and colleagues (Herbst, 2010; Herbst & Chazan, 2003; 2011; Herbst, Nachlieli, & Chazan, 2011) to characterize the obligations to which teachers respond to as they make daily instructional decisions. In the most general terms, the theory states that teachers' instructional decisions are governed by (1) norms that are imposed on a teacher teaching a particular course, (2) obligations that exist towards the profession in which teachers are attached, and (3) personal resources that teachers bring to the environment. These three aspects can be used to understand the rationality of teachers' actions as they teach. Herbst and colleagues argue that there are four types of obligations that instructors respond to with they teach: to the discipline (they are obligated to teach correct mathematics), to the institution (they must abide to the constraints the department gives, such as using the same textbook across sections or giving three major tests to compute students' grades), to the individual students (the instructors must make sure that the lessons reach

individual students, for example by answering questions individual students ask), and to the class as a whole (the instructors must make sure that what she or he does takes into account the different levels of preparation students bring to the class). These professional obligations can explain why teachers may choose, for example, not to answer a student question, even though they may know that it is a good professional practice to acknowledge all students' questions.

This framework is useful to our program of research because we seek to understand the conditions that may impinge on an instructor's decision to use information about their students' understanding to teach differently. We are most interested in learning why the individuals, in their role as instructors, may choose not to use certain information about student learning to plan or modify their teaching. As such, in our work we seek to investigate whether and how tuning into students' thinking, as privileged in K-12 research on teacher knowledge (e.g., Hiebert & Wearne, 1993; Stein, Smith, Henningsen, & Silver, 2000) might reveal teachers' dispositions towards possible instructional changes.

## **Method**

The data for the study were collected between Fall 2010 and Summer 2011 and concerned teaching and learning in three courses, trigonometry, pre-calculus, and calculus taught by two instructors, Elizabeth (trigonometry, pre-calculus) and Emmett (calculus). The teachers selected the mathematical topic to be explored, inverse trigonometric functions, and agreed to (1) collect student data on knowledge of this topic prior to and at the conclusion of teaching a related unit in their courses, (2) allow us to videorecord those lessons in which they taught the topic in the course, (3) allow us to interview their students after implementation of those lessons in order to collect information about students' learning, and (4) meet with us to discuss their views on the findings of the study so that we could collaboratively determine whether and what changes were necessary. At the time of the data collection Elizabeth had seven years of college teaching experience, while Emmett had 16.

### **Data Collection**

We collected three types of data. First we gave paper and pencil tests of knowledge to students from the three courses these two instructors taught (trigonometry, pre-calculus, calculus). Second, we have video recordings of the lessons in which the two instructors taught inverse trigonometric functions in their three courses. Third, we have in depth interviews with students from the three courses, in which we discuss responses to their paper and pencil tests and ask specific questions about what their instructors did in the video recorded lessons (usually conducted the same day of the interview). Fourth we

have several in-depth interviews with the instructors that were used to discuss general aspects of teaching and learning mathematics at community colleges, findings from the paper and pencil tests, findings from the in-depth interviews with students, and possible plans for using the information to modify practice. Forty-five students took the paper and pencil test across the three courses. The test included four questions that sought to determine students' knowledge of inverse trigonometric functions: interpretation of  $\sin^{-1}(0.7)$  and  $\sin^{-1}(2)$ , construction of the graph of  $\sin^{-1}x$  given the graph of  $\sin(x)$ , behavior of the graphs of inverse trigonometric functions, use of the graphs to find specific values of the functions, and interpretation of the intervals in the statement below (hereafter the Identity Question). The Identity was taken from trigonometry textbook that was being used by the instructors:

*Cosine - Inverse Cosine Identities*

$$\cos(\cos^{-1}(x))=x \quad -1 \leq x \leq 1$$

$$\cos^{-1}(\cos(x))=x \quad 0 \leq x \leq \pi$$

**Figure 1.** Stem for the identity question.

We asked the students to (1) explain what the  $x$  in the intervals meant and (2) what they thought would happen if  $x$  were to be taken outside of those intervals. The plan was to collect information on the same content after the unit had been taught, but Elizabeth did not administer the second paper and pencil test. These questions were chosen because they cover the content that teachers must teach. However, they were stated in ways that would require demonstration of understanding of the ideas, as all the questions required students to write explanations for their answers.

We conducted in-depth interviews with 15 students selected from three different achievement brackets, high, average, and low, as assessed by the instructors. There were 5 high-achieving students and three low-achieving students. The rest were average-achieving. We used the listing that the instructors provided us; they relied on the grades students obtained in their first test.

The in-depth interview had three parts. First we asked specific questions about students' responses in their paper and pencil test, to obtain more information about their thinking. Second we inquired about the use of different methods to solve trigonometric problems (e.g., unit circle, triangles, graphs, graphing calculators, etc.) in order to test conjectures about their

understanding of these foundational ideas for trigonometry. Third, we discussed students' interpretation of the identity question (see

Figure 1). Only 10 students answered this question (4 trigonometry, 2 pre-calculus, 4 calculus, mean age = 22 years, sd = 6.57 years). Because of the complex nature of the statement in the Identity Question, we anticipated that students would need to coordinate several foundational notions in order to be able to answer the two prompts successfully (Thompson, Carlson, & Silverman, 2007). As part of the interview, and prior to asking the identity question, we had the students watch Elizabeth's class explanation of the meaning of  $f^{-1}(f(x)) = x$ .

Participating faculty members, Emmett and Elizabeth, were interviewed six times, throughout the year, either individually or in pairs. In the first joint interview, we asked Emmett and Elizabeth to anticipate students' answers to the tasks we had posed on the paper and pencil test and during the interviews. At this point we did not share any of our findings from students' paper and pencil tests and interviews data with the instructors as our goal was to identify of the teachers' own knowledge about their students' understanding of the topics of interest. During an individual interview we asked teachers to comment on the type of explanations they would present in class.

## Data Analysis

**Student paper and pencil test analysis.** The test data were summarized to create reports for each of the participating instructors. The reports indicated how many students answered each of the questions correctly and the common mistakes students had made. For example, in the report for Emmett we stated:

Calculus, Fall 2009. 19 students returned the questionnaire.

### **Question 1:**

Part a asked students to find  $f(g(x))$  given the two functions,  $f(x) = 2x + 1$  and  $g(x) = x^3$ . Two students interpreted the composition as multiplication. The rest of the students obtained  $2x^3 + 1$ . Part b asked students to explain to another student how to find the composition. Students gave explanations consistent with their responses to part a (substitute  $x^3$  into  $x$  in  $f$ , multiply the 2). Part c asked whether  $f(g(x)) = g(f(x))$  and to explain why. One student gave no explanation; another said "I don't know." Three provided a negative answer (not the same) and showed by doing both compositions. Of the students who multiplied one said the answer was the same, the other said that the two functions ( $f$  and  $g$ ) were different, so they were not the same. Three students indicated that in order for the equality to hold, the functions have to be inverses of each other.

**Question 2.**

This question asked to create the graph of  $\sin^{-1}(x)$  given the graph of  $\sin(x)$ . Four students graphed  $-\sin(x)$ . Four students produced a flipped version of a sine over the  $y$ -axis but did not restrict the range or domain. Six created a correct inverse, restricted, but they omitted the values on the axis or did not switch the values on the axis; four produced a correct graph, and one graphed cosecant (incorrectly joining the asymptotes, suggesting a calculator issue). Many answers suggest use of calculator in generating the graph.

**Question 3.**

Part a asked students to identify the errors students made in finding  $\cos(\sin^{-1}(1/2))$ . Three options were given. Most of the students correctly identified the problems students made. Part b asked for a solution with sketches. Five students provided no answer to the question. Four students produced an incorrect response (2 assumed  $\sin^{-1}(x)$  was  $\csc(x)$ , one interpreted  $\sin^{-1}$  as  $\cos$  and composition as multiplication, consistent with the answer to question 1, one interpreted  $\sin^{-1}(x)$  as  $\secant(1/\cos)$ . Nine students provided a correct answer, one of them without explanation (suggesting calculator use), 1 student provided a correct explanation but no solution.

**Summary**

Most students correctly interpret composition of functions. Students with the aid of the calculator are able to produce the graph of the inverse sine function, but most of them make mistakes in labeling the axis. About half of the students can interpret  $\cos(\sin^{-1}(1/2))$  but not many think of a sketch to represent the situation.

**Student Interview analysis.** Similarly to the paper and pencil data, the student responses to the interviews were summarized for each of the questions, across students to gather a general understanding of what appeared to be the common knowledge. The following passage illustrates the summary statement for the interview question that asked for the graph of inverse sine, given the graph of sine:

The student understands the term ‘inverse’ to mean ‘opposite’ in a graphical way, and unsuccessfully tries to use this definition to plot so points of  $y=\arcsin(x)$ . The student’s conception of this graph is logical, but does not comply with the rules of trigonometry... The student does not seem to understand the idea of the domain of a function and its effect on the inverse of the function. He thinks that if the input is outside the domain of a function, then it must be in the input of the inverse function... The student appears to understand that in  $f(g(x))$ , the output of the inner function  $g(x)$  becomes the input of an outer function  $f(x)$ , but doesn’t understand how to relate that knowledge to the range of the outer function. Summary: It is very unclear from this section if the student actually knows what the terms ‘domain’ and ‘range’ mean in a functional/graphical sense...The student does not seem to grasp the connection between the range of  $y=\sin(x)$  and the domain of  $y=\arcsin(x)$ . He looks for patterns in the numbers of the range of each function, but this confuses him more because he cannot find a pattern.

These summaries were compiled for all of the interviewed students to allow for a cross-comparisons by question and by student.

**Teacher Interview analysis.** The focus of our analysis was on what the participating instructors anticipated about students' responses. In this paper we focus only on passages devoted to the Identity Question to identify the professional obligations instructors were responding to. The passages were analyzed thematically (Corbin & Strauss, 2008) to identify teachers' obligations as they commented on the tasks, specifically attending to (1) how they interpreted the statement, in particular focusing on its role in the curriculum (an institutional obligation) and the mathematics (a disciplinary obligation) and (2) how they interpreted students' responses to the Identity Question, as these spoke about their position regarding individual students' understanding (an individual obligation) or what they saw as their roles as teachers in classrooms teaching the ideas related to the Identity Question (an interpersonal obligation).

## Results

We will present first on our findings relative to the analysis of paper and pencil data and then interviews with students, specifically what they understood about inverse trigonometric functions. A discussion of findings regarding the teachers' responses to the students' data will follow.

### Students' Understandings

Our analysis of the students' responses to the paper and pencil test and interview data revealed that the sampled students' understanding of the identity question was based on incomplete conceptions about composition, inverse functions, injective (one to one) functions, domain, range, and angle measures. The interviewed students appeared to experience difficulties with identifying composition as an operation between functions; recognizing that the *identity* for the operation of composition is  $f(x) = x$ , and thus that a bijective (one-to-one and onto) function composed with its inverse results in that identity interpreting inverse of trigonometry functions, in particular the need to restrict the function so that it is one-to-one so and can have an inverse; recognizing the nature of the statement as a statement of truth and the role of the restrictions for making that statement true; managing and understanding the relationship between multiple representations; choosing examples to justify a statement, without attending to the correctness of the example ; and using and interpreting radians, degrees, angles, axis, and periods. The following excerpt, in which one student participant described what each statement in the Identity Question meant to her, illustrates some of these issues. In this description we recognize the difficulties associated with the

identification of composition, restricting input values, and the selection of examples:

STUDENT: This [line 1 in the Identity Question] is saying that the domain for the inverse cosine is in between negative one and one and this [line 2 in the Identity Question] is saying that the domain of the cosine is between zero and pi because this [ $\cos(x)$ ] is the one that we are evaluating first here [line 1] and this [ $\cos(x)^{-1}$ ] is the one that we are evaluating first here [line 2]. The inverse cosine is giving you like one over cosine where  $x$  is the inverse cosine of  $x$  [writes  $1/\cos(x) = \cos^{-1}(x)$ ]. So in doing that you end up with like an indeterminate function if you have your value outside of this [the intervals]... [a value that] does not exist... it gives you error messages because you can't divide by zero.

Notice that in the above statement the student recognizes the importance of the order in which composition is applied: “this is the one we are evaluating first,” by which she means to calculate  $\cos(x)$  first, then take the reciprocal (multiplicative inverse) to apply  $\cos(x)^{-1}$ . But the idea of reciprocal suggests the need to restrict the domain of the function, in this case, restricting  $x$  is associated with avoiding a value—namely 0 in the denominator—that “does not exist... it gives you error messages because you can't divide by zero.”

A casual inspection of the description might convey the notion that the student can identify key markers of the work embedded in the task: in composition one looks at the order of the functions to decide when to apply them and there are domain issues that one must attend to. However, the closer look also suggests that the notion of inverse for composition is meshed with notions of multiplicative inverses (or reciprocals), driven by the notation used ( $\cos^{-1}(x)$ ). Using  $\arccos(x)$  might have prompted a different response. This suggests to us that the ideas of composition of functions, as an operation was weakly understood by the student. We anticipated that these difficulties and the possible sources (e.g., the notation for inverses under composition and of multiplicative inverses is the same in texts,  $-1$ , but they refer to different objects) would also give instructors a space for suggesting instructional changes. We had anticipated that the students' misuse of terminology and procedures would prompt some discussion among the participating teachers.

### Teacher Interviews

Recall that we interviewed the instructors twice, once before the interviews with the students to inquire about the potential difficulties their students would have had with the content under consideration, and once with the summaries of the students' responses to the paper and pencil tests and the prompts used during the interviews with them. During each of the interviews we asked the teachers to answer the identity question and they produced explanations that addressed four mathematical foci: (1) the identity is a particular case of  $f^{-1}(f(x)) = x$ ; (2) trigonometric functions are periodic and

not one-to-one; therefore one must restrict the functions to obtain inverses; (3) when dealing with inverses, “one function undoes the other,” which is why  $x$  is obtained; and (4) the different values in the two intervals stem from the different order in which the functions are composed. Neither of the instructors however, explicitly indicated that the restrictions in each line operated differently: While the restriction in the first line is needed to ensure that the inverse function can be calculated, the restriction in the second line is needed in order to ensure that the equality holds ( $\cos(x)$  takes all real values,  $\cos(2)$  is a number between -1 and 1, and therefore  $\cos^{-1}(\cos(2))$  exists, but it is not equal to 2; so one has to exclude numbers larger than 1 and smaller than -1 for making the statement true):

Elizabeth: [we restricted it because] we wanted all of the fabulous function properties. So we had to make sure now because of the domain and range being switched as long as all the range elements are there, then when we switch them all the domain will be complete for our inverse. So this is just a restricted domain. I would, again, refer to a picture [of the unit circle], right?  
 Emmett: The inner function here [second line] is cosine. Although  $x$  can be anything from minus infinity to plus infinity it won't work because it's not a one-to-one function. Its inverse is not a function.

In our sample of 10 students who answered the Identity Question, only one student had a similar realization than that of the teachers. Teachers' anticipations about the mistakes students would make were ambiguous,

Elizabeth: [students would say] this is going to be really hard to remember (...) this is confusing to remember.' And I tell them absolutely, yes, I agree. It is confusing to remember but it's easy to know (...) I think that what they're probably going to think is that all the  $x$ s that come out are going to be between zero and pi (...) I think to them they're going to look at this as 'oh this is just an observation' right?  $x$  is just going to come out like this. That would be my guess. I'm not positive.

Elizabeth's statements suggest that she considered students would read the intervals as the range of the statement rather than as the domain of the inverse functions. When pressed for elaboration Elizabeth indicated that she believed the students would see the intervals as “outcomes” of the statements. Emmett thought that students would have no problems with the first statement. He did raise concerns about the second statement:

Emmett: So they might say, 'Why do we even have this restriction?' The second one. The first one I think they'd understand (Elizabeth: Yeah) because it's an inverse cosine. (Elizabeth: I agree) minus one-to-one no issues, but the second one they might say well, why? We tried some examples. Then we tried one that worked. I tried one in that range but I tried one not in the range. And then I asked them 'why did the calculator give you a different number?' And that different number that the calculator gives them

is actually within this range. So the outcome is always going to be between zero and pi and it just has to do with one-on-one functions.

Emmett said that he had worked this out with his students, picking values outside the intervals and asking the students to see what would happen, and at the end he indicates that the reason for the restriction is that the function has to be one-to-one.

Our interpretations suggest that the teachers' knowledge of mathematics for teaching (Ball, Thames, & Phelps, 2008) regarding the sources of the students' mistakes is not well developed among the participating community college instructors. This is not surprising given the limited pedagogical preparation that college math instructors have and the scarcity of research concerning their pedagogy.

During the interviews we also presented the teachers with summaries of data from their students' written paper and pencil tests and the student interviews. Both teachers were surprised to read the students' responses and attempted to explain the sources of such conceptions. The teachers attributed the students' results to their exposure to curriculum which lacked transparency in the notation, and its reliance on technology:

Emmett: even within the course, I do not need to go back to limits all the time. I think with inverses is tricky. Because we use -1 for many things, exponent and inverse composition.

Elizabeth: The trig book we have now... they like to get them pushing buttons on page one (Emmett: Mm-hmm) inverse tangent buttons. I never assign those problems and I tell them don't even read that stuff because I do not want you thinking that you know something about an inverse trig function without the full story. Because it is not a button pushing game. So I disagree with our book.

Both teachers also raised issues regarding the influence of the students' prior knowledge and their habits regarding mathematical work:

Emmett: Because of the foundations that they don't have. I am always telling them it's not just okay to know how to do the problem. You have to do it correctly, also. That means avoid algebraic mistakes as much as you can.

When asked for specific actions they could take to address the conceptions unveiled through student data. In response, the instructors suggested concrete actions they *had* taken in the past, rather than devising new models for future work:

Elizabeth: what you would do is ask a question: 'how is your -1 different in these two expressions? Tell me the difference between these two identical representations.' That could be a legal question.

Emmett: With my students we already did that. I mean we picked an  $x$  outside of this interval and it just doesn't work. The second [line 2], they get something but what they will get is something in this interval. Always. The first [line 1] doesn't work out at all you just get, it just shouldn't. Because if you pick, let's say inverse cosine of 2 it's just not going to work. However cosine of  $2\pi$  still exists but the inverse cosine of  $2\pi$  isn't  $2\pi$ . The inverse cosine of cosine  $2\pi$  isn't  $2\pi$  because that's outside the range. So it takes you back to zero. I've even tried examples and I ask them this question. I said: 'if you take a number and you apply one function to it and then its inverse it should get you back to that number.'

When asked about students' difficulties with composition of functions, a topic covered in an earlier pre-requisite course, both instructors indicated that the textbook the department had adopted did not contain an in-depth discussion of that particular topic, thus again, a curriculum justification for the students' difficulties was provided. Both instructors indicated that the recent change of textbooks (one single textbook for college algebra and pre-calculus) and a course organization prompted by the change (to eliminate repetition of topics) would better address these misconceptions in the future.

### **Discussion and Conclusion**

In this study we sought to investigate how community college instructors teaching trigonometry interpreted information about their students' understanding of inverse trigonometric functions and how they used such information to plan their lessons. We learned that the participating instructors had limited knowledge of how their students thought and understood foundational mathematical concepts—angles, functions, composition, graphs, and inverse functions. They attributed the students' misunderstandings to two sources: curricular (how topics are presented and organized), and cognitive (inadequate student prior knowledge).

We found also that the participating instructors could not offer suggestions on how the information on students' knowledge could be used to plan their lessons. The instructors responded with ideas they had previously used rather than designating new steps they would need to take in the future when teaching the same topic. This suggests that they saw the curriculum and the students' prior knowledge as stronger forces that operate outside their control. This perception revealed limited agency on the part of the instructors over their professional work. They perceived their responsibility to consist of making sure that materials were presented in a common syllabus—a clearly outlined institutional obligation, and that it was the students' responsibility to learn the material by attending the lectures, doing the homework, and taking exams. These views pronounce the teachers' recognition of their obligations towards institutional requirements (cover the syllabus) rather than disciplinary, individual, or interpersonal obligations

These results suggest the need to further investigate what should be the nature of faculty development in this setting, given that teacher knowledge for teaching appears to be weak. It might be possible that by augmenting teacher knowledge, the faculty might be able to start shifting their attention to their other obligations as they reason through instruction.

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