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Algebraic Modeling in a Proof Context

Mara V. Martinez
University of Illinois at Chicago

Bárbara M. Brizuela
Tufts University

In this paper, we propose a refinement of Chevallard's algebraic modeling while still fully agreeing with his position that it is a key process in knowing mathematics. In doing so, based on our empirical study, we claim that: 1) there are two other stages in the algebraic modeling process; 2) it is non-linear; and 3) partial models are constructed as well. Research on Chevallard's modeling perspective is relevant to the international mathematics education community, given his specific emphasis on algebra. Each one of the above claims will be illustrated through episodes from a classroom intervention with nine 9th/10th grade students who participated in a teaching experiment.

Keywords*:* proof, algebra, modeling, conjectures, mathematical models

Mathematical modeling plays a central role in conceptualizing learners' mathematical understanding and has been studied within varied research traditions (e.g., Freudenthal ,1973; Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). However, so far, the majority of empirical studies focus on modeling in mathematics in general, without considering specific domains within mathematics (e.g., geometry or algebra) and how the specific domain context may impact the descriptions of the modeling process. There are a few exceptions (e.g., Chevallard, 1989; Hanna & Jahnke, 2007); one of them is the theoretical framework developed by Chevallard (1989), which although specifically focused on a specific domain (i.e., algebra), still lacks an empirical basis given that it was conceived as an epistemological product as opposed to a description of how students model. In his framework, algebra is seen as the modeling tool, and the modeling process is described in terms of algebraic objects and operations (i.e., variables, parameters, and relations among them). It is the framework's specificity that makes it relevant for conceptualizing the use of algebra as a modeling tool in the construction of algebraic proofs. That is, we argue that other modeling frameworks are too general to prove truly useful for our work in algebra. In spite of this exception, Chevallard's (1989) definition of modeling has not been studied empirically. Thus, the study we describe in this paper empirically examines Chevallard's modeling process in the context of algebraic proof.

Consequently, our overarching research question is: How does our empirical study inform Chevallard's modeling process? More specifically, how does students' modeling relate to Chevallard's stages? Does students' modeling process follow the order of stages established by Chevallard? If it does not, how does it differ? And, what are students' processes at the interior of each stage?

We put forth three claims regarding a revised process based on the results of our empirical study: (1) there are two other stages: "Interpretation of the Problem" and "Production of Competing Conjectures"; (2) algebraic modeling is a non-linear process; (3) partial models are constructed during the process as well. While some of the proposed refinements are not necessarily novel within other research traditions that have focused on mathematical modeling in general (e.g., Freudenthal, 1973; Hanna & Jahnke, 2007; Lesh et al., 2003), we argue that Chevallard's perspective is relevant to the international mathematics education community—and therefore merits special attention,—given his emphasis on algebra. Acknowledging these potential refinements is relevant in bringing closer together bodies of research in mathematics education that have remained mostly isolated (e.g., Chevallard, 1989; Freudenthal ,1973; Lesh et al., 2003).

We will provide evidence for the proposed revision through an in-depth analysis of a triad of 9th/10th grade students who worked with the first author of this paper during a teaching experiment that focused on students' use of algebra as a modeling tool to prove.

Algebra as a Modeling Tool

In Chevallard's (1985, 1989) perspective, modeling plays a key role in knowing mathematics. In our work (Martinez, 2011; Martinez, Castro Superfine, 2012; Martinez, Brizuela and Castro Superfine, 2011) that focuses on algebraic proof, his theoretical perspective has been valuable to conceptualize the stages that students might go through when modeling with algebra. Chevallard (1989) described the three stages of the algebraic modeling process in the following way:

(1) We define the system that we want to study by identifying the pertinent aspects in relation to the study of the system that we want to carry out, in other words, the set of variables through which we decide to cut off from reality the domain to be studied … (2) Now we build a model by establishing a certain number of relations R, R', R'', etc., among the variables chosen in the first stage, the model of the systems to study is the set of these relations. (3) We 'work' the model obtained through stages 1-2, with the goal of producing knowledge of the studied system, knowledge that manifests itself by new relations among the variables of the system. (p. 53.)

While other researchers have certainly already studied mathematical modeling (e.g., Freudenthal, 1973, 1983; Hanna & Jahnke, 2007; Lesh et al., 2003), the existing descriptions are insufficient to account for the complexity and specificity of mathematical modeling in different domains within mathematics, such as algebra.

Lesh and his colleagues have largely studied modeling with an emphasis on "real world" contexts (e.g., Lesh, et al., 2003; Lesh & Doerr, 2003; Lesh, Lester, & Hjalmarson, 2003; Lesh & Zawojewski, 2007). Within this tradition the development of models is conceived as part of problem solving. While Lesh's proposal has advanced our understanding of the modeling process, from our perspective it lacks exactly what Chevallard's description of the process has to offer: a focus on algebra as a modeling tool.

Hanna and Jahnke (2007) also investigate modeling within non-mathematical contexts. They use arguments from physics as a method to build an explanatory proof. According to Hanna and Jahnke (2007), "modeling often has to do with creating a non-physical representation of a physical system" (p. 147) and they relate their approach to "reality related proofs." However, of special importance for our study is that they view modeling and proof as being inextricably linked and as having complementary roles. Taken together, Chevallard's (1989) modeling process provides us with a definition specific to algebra while Lesh et al. (2003) underline the non-linearity of the process, and Hanna and Hanke (2007) emphasize the inter-relation between proving and modeling. More specifically, Chevallard's modeling process is of special interest to our work on modeling and algebraic proof given that it conceptualizes algebra as a modeling tool and defines modeling in terms of algebraic objects and operations (e.g., variables, parameters, and relations among them).

However, given that Chevallard's framework is theoretical and with an epistemological focus, it remains to be studied empirically. Thus, the study presented here examines Chevallard's framework through the lens of students' use of algebra as a modeling tool to prove.

Proof Framework

In our work we develop and study problem situations that can foster students' learning of algebraic proof (Martinez, 2011; Martinez, & Castro Superfine, 2012; Martinez, Brizuela, & Castro Superfine, 2011). To design these situations we have focused on the use of algebra as a modeling tool. Regarding proof, our framework has been conceptualized by bringing together the work of Balacheff (1982, 1988); Hanna (1990); Arsac and his colleagues (1992); and Boero (2006). From the work of Balacheff (1982, 1988), we build on the idea that proof is an explanation that is accepted by a community at any given time.

We also drew from Hanna's (1990) distinction between proofs that (just) prove and proofs that (also) explain. The first kind just establishes the validity of a mathematical statement. The second kind, in addition to proving, reveals and makes use of the mathematical ideas that motivate it. In a similar vein, Arsac et al. (1992) proposed three roles for proofs as part of an instructional task: to understand why and/or to know, to decide the truth-value of a conjecture, and to convince oneself or someone else. Consequently, in the teaching experiment we adopted this broader view of the role of proof that goes beyond establishing the truth-value of a statement. In fact, we adopted a design principle according to which a proof has the potential to help students understand why a specific phenomenon happens. Therefore, in the algebra problems we focused on in the teaching experiment, the role of proof was not only to establish the truth-value of a statement but also to foster students' understanding of why a specific mathematical phenomenon happens.

Another critical aspect of our work, following Boero (2006), is that conjecturing and proving are inter-related crucial mechanisms for generating mathematical knowledge. Therefore, they must be taught jointly. Indeed, in the Calendar algebra problems used in the teaching experiment, students were not provided with the conjecture to prove. As part of the problem they had to generate their own conjectures and then prove them.

Therefore, this research on proof, together with the aforementioned research on modeling, frames our current study that aims to empirically examine Chevallard's modeling process.

Methodology

Participants

In this paper, we describe and analyze the work of three 9th/10th grade students who were 14 to 15 years old (i.e., Abbie, Desiree, & Grace) and participated in a teaching experiment, led by the first author of this paper, in which a total of nine 9th/10th graders took part, at a public charter school in Boston, Massachusetts in the United States.

Procedure

Fifteen one-hour lessons were held once a week. These lessons were part of the regular school schedule but not part of the students' regular mathematics classes. In this paper we will report on data collected during Lessons 1 and 2, in which students focused on Problem 1 (Parts 1 and 2), which were implemented during the first half of the intervention during which students made use of multiple variables, and equivalent expressions to prove their own conjectures.

Problem 1

Part 1: Consider a square of two by two formed by the days of a certain month, as shown below. For example, a square of two by two can be 1 2
8 9

These squares will be called 2x2 calendar squares. Calculate the difference between the products of the numbers in the extremes of the diagonals. Find the 2x2 calendar square that gives the biggest outcome. You may use any month of any year that you want.

Part 2: Show and explain why your conjecture is true always.

Figure 1. Problem 1 from the calendar algebra teaching experiment.

Students were provided with calendars corresponding to years 2005-2008 accompanying Problem 1-Part 1 (see Figure 1). As a result of students' work on Part 1, they implicitly had to analyze the nature of the outcome of the described calculation (i.e., subtraction of the cross product). It was expected that students would anticipate some kind of variation in the outcome in relation to the set of days where the operator is applied and that students would find out, through exploration, that the same outcome is always obtained (i.e., -7), no matter where (i.e., square location within a month, across months, and across years) they apply the operator. The ultimate educational goal of Part 1 of Problem 1 was to get students to produce conjectures about the behaviour of the outcome (i.e., subtraction of the cross product) as it relates to the location of the square in the calendar. After each group of students produced their conjectures, as part of Part 2 of Problem 1, students had to gather evidence to show that their conjecture was true. The challenge for the students was to find out why this happens, and whether this is "always" going to be the case. At this stage in the problem, from a mathematical point of view, algebra becomes a tool to solve the problem. Thus, one of the challenges in Problem 1 is to show the limitations of using a non-exhaustive finite set of examples to prove that the proposition is true, and to encourage students to use algebra as a tool that allows them to express all cases using a unique expression.

Findings

Claim 1: There are two other stages in the algebraic modeling process

Interpretation of the Problem. Given that our interest in mathematical modeling, specifically the use of algebra as a modeling tool to prove, is educational, it is crucial to include a stage that accounts for students' processes when they are mostly focused on understanding the statement of the problem, which does not exclude questions regarding the statement of the problem from re-appearing once students start to "solve" the problem (and which would relate to the non-linear nature of the modeling process). In Abbie, Desiree, and Grace's group, the first five minutes of small group work were characterized by trying to understand the meaning of the statement of the problem. Among the issues that surfaced during this interpretation stage were: potential strategies to solve the problem; reaching an agreement regarding the operations involved; the nature of the numbers considered (negative, positive, absolute value); and the delimitation of the domain to study. After these initial minutes, the students reached an agreement regarding these issues and their main focus shifted towards the production of conjectures regarding the outcome.

The appearance of this stage, we hypothesize, may be related to the fact that the problems were open ended. However, as in any problem solving activity, we think it is important to bear in mind that students will go through a stage when they make sense of the statement of the problem. The knowledge that students construct during this stage will impact their subsequent approach to the problem. Some issues appeared in this stage that had to be negotiated, for instance, whether months could be overlapped, whether empty cells are counted as zero, and the order of the subtraction of the cross products. These are interpretation issues that define what (i.e., the phenomenon) will be studied mathematically.

Production of Competing Conjectures. This stage is characterized by students' production of conjectures. As part of their investigation students pondered how to obtain the largest outcome. As a result, students studied the nature of the dependence between the outcome and different aspects of the situation. For instance, they investigated how the position of the square within a month would impact the outcome.

During this stage, students constructed five different conjectures (see Figure 2). Students also explored the possibility of each of the conjectures being true. By the end of this stage, students were relatively certain of the truth-value of one of the conjectures (i.e., the outcome is always -7). Students employed a significant amount of time figuring out the behavior of the outcome and its dependence on/independence of different elements of the context.

In doing so, students tested their conjectures by analyzing different cases. Indeed, we observed that what has to be proved is not obtained through a straightforward process, but rather through students' analysis of the likelihood of the truth-value of each of the competing conjectures. The first conjecture, C1, linked the numeric outcome to the square's position at the beginning of the month, while the second, C2, is a claim about the outcome's independence of the month that it was placed in. After abandoning C1 and C2, a third conjecture, C3, was momentarily embraced: possibility of the outcome's dependence on the month where it is placed, which is in contradiction with C2.

Right after, students produced a fourth conjecture, C4. C4 seems to be a refinement of C3, given that it relates the potential variation of the outcome with what day of the week is the first day of the month. Students identify not only what month the square is placed in but also what day of the week is the first day in that month as potential sources for the variation in the outcome. Students hypothesize that a potential source of variation in the outcome is the relative arrangement of the days. To test their conjecture, they tried months that differ in what day of the week is the first day of the month, always arriving at the same outcome, -7. As a consequence of trying all imaginable unfavorable scenarios and invariably obtaining -7, students produced their fifth and concluding conjecture (C5) stating that the outcome is always -7 regardless of different arrangements of numbers, months, and first days of the month.

Claim 2: Algebraic modeling is a non-linear process

We claim that the algebraic modeling process is non-linear: stages can occur in non-consecutive order, and therefore does not necessarily follow the path as described by Chevallard (1985, 1989). This result concurs with previous studies within other frameworks (e.g., Hanna & Jahnke, 2007; Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). As illustrated in Table 1, students already started analyzing some relations among variables (i.e., “Establishing Relationships Among Variables” stage) in the “Interpretation of the Problem” stage. In our analysis, it is possible to identify a main focus to students’ work during a certain period of time. For instance, during the first five minutes, as the students were trying to understand the statement of the problem, elements from other modeling stages appeared. During “Interpretation of the Problem,” Abbie noticed the relationships between elements in the corner of the diagonal of the square (i.e., “Establishing Relationships Among Variables”). At that moment, her group did not elaborate on her comment until later when they were mainly producing conjectures; more specifically, until the moment of writing the expression to prove their concluding conjecture, illustrating that stages do not necessarily happen in a fixed order.

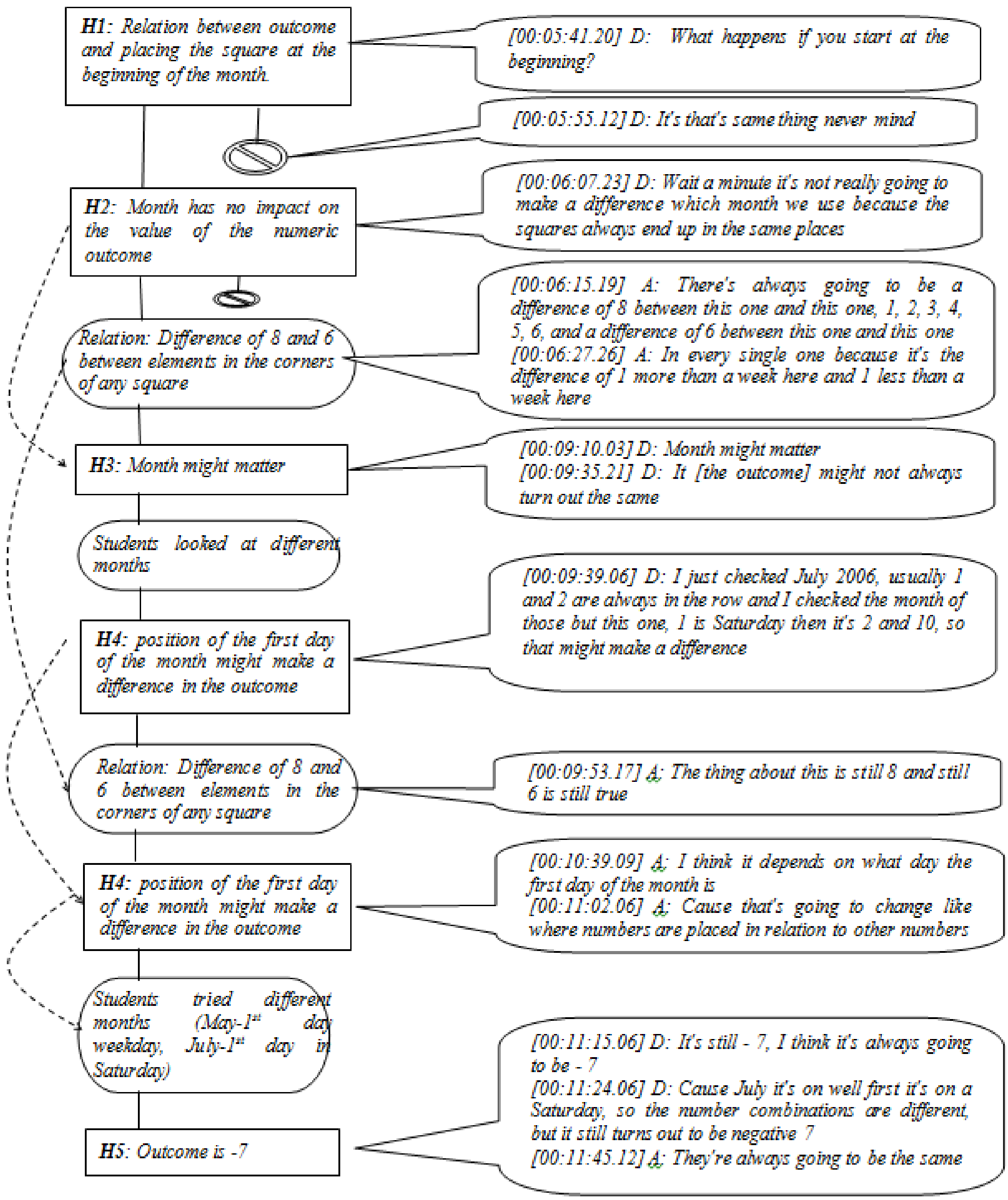

Figure 2. *Proposed production of competing conjectures stage. Conjectures are represented on the left in order of appearance. On the right, excerpts of students' discussion along with the time code and the student's name initial.*

Table 1
Illustration of Non-Linearity of The Modeling Process

Transcript	Stage	Significance
[00:05:41.20] and [00:06:07.23] Production of C1 and C2.	Production of Competing Conjectures	New proposed stage.
[00:06:15.19] A: There's always going to be a difference of 8 between this one and this one, 1, 2, 3, 4, 5, 6, and a difference of 6 between this one and this one.	Establishing Relationships Among Variables	Chevallard proposed this as the final stage in the modeling process.
[00:09:10.03] Production of C3.	Production of Competing Conjectures	New proposed stage.
[00:09:53.17] A: The thing about this is still 8 and still 6 is still true.	Establishing Relationships Among Variables	Chevallard proposed this as the final stage in the modeling process.
[00:09:39.06] Production of C4.	Production of Competing Conjectures	New proposed stage.

Claim 3: Partial models within stages of the mathematical modeling process.

As aforementioned, even though Chevallard's mathematical modeling has advanced the field's understanding of algebra as a modeling tool, it does not account for the complexity involved at the interior of the modeling stages. For instance, within Chevallard's "Establishing Relationships Among Variables and Parameters" (i.e., third stage), we have found that students produce partial models of the situation under consideration. A partial model is a model that includes only some variables, parameters, or relations among them. In contrast, a complete model is what we would traditionally call model, and it includes all variables, parameters, and relations that are relevant to study the situation under consideration. Students' partial models show that stages are not a one-step process. In fact, partial models and their later transformation into a complete model not only illustrate the complexity at the interior of the stages but also suggest its cyclic nature.

For instance, building on the relations identified when they first encountered the problem, Abbie proposed to write with "algebra" the following (see Figure 3): "All right, so this is what I have: 'x' times 'x minus 8' [referring to the product between the elements in one of the diagonals in this square $\begin{matrix} x-8 & y-6 \\ y & x \end{matrix}$], minus 'y' times 'y minus 6' [referring to the product between the elements in the other diagonal in the same square $\begin{matrix} x-8 & y-6 \\ y & x \end{matrix}$]. That's basically what we're doing, yes."

The square $\begin{matrix} x-8 & y-6 \\ y & x \end{matrix}$ is a partial model of the situation. In fact, the relation between x and y was not included as part of it. This shows that the third stage "Establishing Relationships Among Variables and Parameters" does not fully occur in one step. On the contrary, it occurs partially as manifested by students' construction of partial models.
After producing the partial model, students continued to check whether or not the algebraic expression was correct; in order to do that, they were going to replace the letters, at least initially, with random numbers, as shown in the exchange below:

Desiree: Just plug in random numbers?
Abbie: Yeah.
Abbie: 12 times 12 minus 8 minus [referring to the replacement of x=12 in the square $\begin{matrix} x-8 & y-6 \\ y & x \end{matrix}$, yielding $\begin{matrix} 12-8 & y-6 \\ y & 12 \end{matrix}$], oh wait but 'y' and 'x' also are related because 'y' is one less than 'x.' OK, so we can go 'x minus 1, times, x minus 1 minus 6'.

At this point, Abbie realized that the variables x and y are, in fact, related (i.e., y is one less than x); thus once a value is assigned to x, y's value is one less than x's.

Desiree: OK, what?
Abbie: So 'y' and 'x' aren't like random numbers cause like this wouldn't be 17 and that wouldn't be 2 [referring to the numeric values of x and y at the bottom of the square $\begin{matrix} \cdots & \cdots \\ 2 & 17 \end{matrix}$, that's [referring to the y in $\begin{matrix} x-8 & y-6 \\ y & x \end{matrix}$] always going to be 1 less than that [referring to the x in $\begin{matrix} x-8 & y-6 \\ y & x \end{matrix}$], so we can say everything in terms of this number right here [referring to the x in $\begin{matrix} x-8 & y-6 \\ y & x \end{matrix}$].

When checking the expression, Abbie realized that y and x are in fact related: "'y' and 'x' aren't like random numbers … so we can say everything in terms of this number right here [referring to the x in $\begin{matrix} x-8 & y-6 \\ y & x \end{matrix}$]. Abbie's insight allows them to formulate a complete mathematical model, including

one independent variable x and three dependent variables x-1, x-8, and (x-1)-6 with explicit relationships among the different variables (see Figure 3).
In this case, students came to realize that there were two relationships that they had overlooked; the relationship between x and y in the square $\begin{matrix} x-8 & y-6 \\ y & x \end{matrix}$, and, consequently, the entailed relationship between x and y-6 in the square $\begin{matrix} x-8 & y-6 \\ y & x \end{matrix}$. Therefore, they "went back" to incorporate these "unnoticed" relationships in the partial model producing, as a result, the complete model.

In summary, partial models illustrate the complexity of the process within the modeling stages. Students' production of the complete model as a refinement of the partial model once again suggests the cyclic nature of the stages.

$$\begin{matrix} x-8 & y-6 \\ y & x \end{matrix} \quad \xrightarrow{y=x-1} \quad \begin{matrix} x-8 & (x-1)-6 \\ x-1 & x \end{matrix}$$

Figure 3. **Students' partial model (left) and complete model (right).**

Concluding Remarks

In this paper, we have proposed a refinement of Chevallard's algebraic modeling process. This refinement was informed by findings from our empirical study of students' modeling process when using algebra as a tool to prove. Each one of the claims below has been illustrated through episodes from a classroom intervention with nine 9th/10th grade students who participated in a teaching experiment. One of the goals of the classroom intervention was to provide high school students with the opportunity to use algebra as a modeling tool to prove.

First, we proposed the inclusion of at least two others stages, namely: "Interpretation of the Problem" and "Production of Competing Conjectures."
Regarding the first proposed stage, we showed elements of the context of the problem that were subject to interpretation (e.g., overlapping months, order of calculation, etc.). This is relevant given that students' ultimate approach to the problem depends on the meaning constructed during this "first" stage. Indeed, students did not go directly to "identify variables and parameters," that is, Chevallard's first stage. Instead, students first made sense of the problem (e.g., considering both positive and negative numbers as a possible numeric value for the outcome). In his problem-solving framework, Polya (1945) referred to this stage as the first principle: understand the problem. Emphasizing this stage further seems so obvious that it is often not even mentioned; yet students are often stymied in their efforts to solve problems

simply because they don't understand it fully, in part, or have an alternative understanding.

Regarding the second stage, we have illustrated how the final conjecture emerged (i.e., C5, “outcome is always -7”) as a result of students’ conjecturing process by producing a diverse set of competing conjectures (e.g., C4, “the outcome may depend on the day of the week in which the month begins”), and analyzing their likely truth-value (e.g., by trying different examples).

In comparing the generality of these proposed two stages and their potential emergence in other modeling situations, the first one seems applicable to a wider variety of situations than the second stage. The emergence of the “Production of Competing Conjectures” stage is related to the proving nature of the mathematical task we focused on.

Second, we provided evidence illustrating the non-linearity (e.g., stages do not happen in a fixed order) of the mathematical modeling process as described by Chevallard. In particular, we showed an example of how students cycled through the “Production of Competing Conjectures” and the “Establishing Relationships Among Variables” stages. Although this concurs with findings reported by researchers from other frameworks (e.g., Hanna & Jahnke, 2007; Lesh et al., 2003), our study addressed the issue of linearity/non-linearity taking into account the stages and their order as proposed by Chevallard.

Lastly, and given that Chevallard’s mathematical modeling process does not account for the complexity involved at the interior of each of the modeling stages, we introduced the idea of partial model to account for models that include some (but not all) variables, parameters, or relations among them. In fact, these two types of models illustrate the process nature and complexity at the interior of the stages in the modeling process. As aforementioned, modeling has been studied within different research traditions (e.g., Freudenthal ,1973; Lesh, et al., 2003). However, the vast majority focus on modeling in mathematics without paying attention to whether, and potentially how, modeling is shaped in specific domains within mathematics. There are a few exceptions (e.g., Chevallard,1989; Hannah & Jahnke, 2007). Still, we lack thorough studies regarding to what extent specific domains within mathematics affect the modeling process. In this paper, we showed how modeling manifested itself in the context of algebraic proof. By doing so, we contributed to advance the understanding of algebraic modeling.

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Authors:

Mara V. Martinez
University of Illinois at Chicago
Email: maravmar@uic.edu

Bárbara M. Brizuela
Tufts University
Email: barbara.brizuela@tufts.edu