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A Different Content and Scope for School Arithmetic

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Modern school arithmetic has evolved from methods that were used to teach 19th century accountants. Today its scope is broader, but the sequence of topics remains almost the same. The goal of teaching arithmetic has changed from achieving speed and accuracy in written computation to understanding mathematical principles and the structure of algorithms. Attempts have been made to reach this goal just by changing pedagogy, but students are still required to master skills of written computation. To carry out written computation, students need good penmanship and instant recall of arithmetic "facts", which is challenging to many, especially those who speak different languages at home and at school. We present a different approach to teaching elementary arithmetic based, not on written computations, but on algorithms executed on counting boards designed on principles invented by John Napier. This approach suggests a radically different sequence of topics and some changes in content, and teaches arithmetic as a tool for solving practical problems in low- and high-tech environments.

Keywords: Arithmetical algorithms, counting boards, scope and sequence.

The history of mathematical notation shows that writing was rarely used as a tool for computation e.g., Chrisomalis (2010), but that computation was more often done with the help of low-tech devices (counting boards). During the last 50 years, modern computing devices have rendered paper and pencil calculations obsolete, but students in early grades need a large amount of hands-on experience to learn the concepts of arithmetic. Written computation is still the main teaching tool for learning those concepts. A variety of other representations of numbers attempt to make arithmetic concepts more concrete, and among them the use of a number line is most often recommended by the Common Core Mathematics Standards (National Governors Association et al., 2009).

John Napier (1617; 1990) in his *Rabdology,* designed a counting board where values of locations in each column (called a rod) form a geometric progression with factor 2, and he showed that most arithmetic computations that were known during his time (including computations of roots) could be carried out on his board with the help of just one simple set of regrouping rules. This work was completely ignored, probably because he used only binary numbers on his boards (Gardner, 1986). But it is sufficient to keep geometric progressions only in columns, which gives one all the advantages of Napier's invention. When one puts powers of ten in one row of a board, all calculations are carried out in decimal notation. Thus, such "decimal" boards are sufficient to carry out all written computations that are currently taught in elementary and middle school. Also, because computation on boards is "hands-on" and the rules of regrouping are easy to justify, teaching in this way can make unnecessary many other teaching aids that are used in early grades. Changing a basic tool for computing from paper and pencil to counting boards could bring significant changes in the order of topics and the scope of the material that is covered. Such a change is too radical to be feasible, so we treat counting boards as another teaching aid, and try to show how counting boards that are constructed on Napier's principles can be used for teaching specific topics within existing curricula.

The purpose of this paper is to describe the construction and properties of counting boards designed on John Napier's principles that provide an alternative way of teaching arithmetic. We do not address pedagogical issues that deal with methods of teaching the use of counting boards, because we do not have enough classroom data to provide any reliable recommendations.

Boards and Their Mathematical Content

Rods and Boards

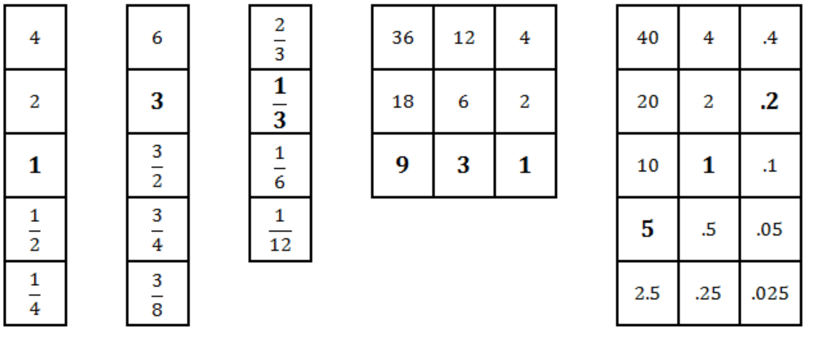


Figure 1. *Examples of empty rods and boards showing values of locations (Roots of rods are shown in bold print).*

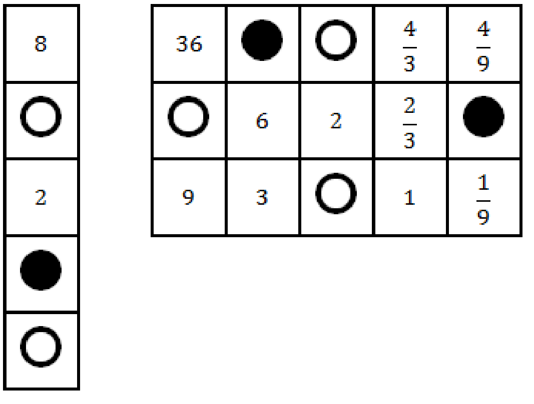
Each board is a list of rods (see Figure 1). A rod is a list of locations that have values that form a geometric progression $a_1, a_2, \ldots, a_n$, with factor 2, where $n > 1$, such that $a_1 = r*2^e$ where r, called a root, is either an odd positive integer or its reciprocal, and e is an integer (that can be positive, negative, or zero).

The length n of a rod is not theoretically important, because any rod may be extended up and down whenever it is needed. In practice, so far, we have never needed to use rods that have more than 10 locations, but we can imagine rods of any length.

Any board has to contain a rod with root 1, but how other rods are aligned may vary, so the same set of rods can form different boards.

Numbers are represented by configurations of tokens on a board. Each token has its absolute value v, and a local value on a board that is the product of v and the value of the location it occupies. The value of a board is the sum of all local values of the tokens.

It is sufficient to have tokens with only two values, 1 and -1. We use white tokens to represent 1 and red tokens to represent -1, as shown in Figure 2.



$4 - 1 + \frac{1}{2} = 3.5$ $\qquad$ $18 - 12 + 4 + 1 - \frac{2}{9} = 10 + \frac{7}{9}$

Figure 2. *Representing numbers on a board (A white token has the value 1 and a red token, shown in black, has the value -1).*

Numbers Represented on One Rod

(i) When r =1, then the numbers that are represented on one rod are integers and also fractions that have powers of two as denominators (binary fractions). Technically it is a small ring of integers in which only the prime number 2 has a reciprocal.

(ii) When r is an odd positive integer, then integers represented on the rod are limited to those that are divisible by r. So they form a subset of the numbers described in (i).

(iii) When r is a reciprocal of an odd positive integer d > 1 ($r = d^{-1}$), then the numbers that are represented consist of all improper fractions with denominators that are factors of d. This set always contains as a subset the set described in (i).

Numbers Represented on One Board

Let m be the least common multiple of all numbers d whose reciprocals are roots of the rods, $r = d^{-1}$.

Theorem. The set of numbers on a board consists of all improper fractions whose denominators are factors of m, multiplied by any power of 2.

...	...	...
2	$\frac{2}{15}$	$\frac{2}{21}$
1	$\frac{1}{15}$	$\frac{1}{21}$
$\frac{1}{2}$	$\frac{1}{30}$	$\frac{1}{42}$
...	...	...

Figure 3. *A board with 3 extendable rods with roots 1, 15^{-1}, and 21^{-1}.*

Here, m = lcm (1, 15, 21) = 105. So the set of numbers represented on this board consists of all numbers $i*105^{-1}*2^{j}$, where i and j are integers (see Figure 3).

Appending New Rods to a Board

In practice we have never used a board that has more than 15 rods, but there is no limit on the number of rods on boards that we may imagine. But we put some restrictions on adding new rods, which we explain below. Any rod with the integer root r can be added to a board. But if a root is the reciprocal of an integer, $r = d^{-1}$, it can be added only if each prime factor p of d is also a factor of d’, where $r' = d'^{-1}$ is already a root of a rod on the board. In the example shown in Figure 4, one added rod has root 1/15. It is allowed because 1/5 and 1/3 are already roots of other rods.

An original board

1	$\frac{1}{3}$	$\frac{1}{5}$
$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{10}$
$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{20}$

and its extension

30	10	6	2	$\frac{2}{3}$	$\frac{2}{5}$	$\frac{2}{15}$
15	5	3	1	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{15}$
7.5	2.5	1.5	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{10}$	$\frac{1}{30}$
3.25	1.25	.75	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{20}$	$\frac{1}{60}$

Figure 4. *A typical case of appending rods to a board.*

There is a reason for making this restriction. As we will see below, for each board we have a set of rules for changing the patterns of tokens, which we call "regrouping rules". As it happens, we need exactly one rule for each reciprocal of a prime number. But finding these rules
requires factoring a number whose reciprocal is a root. But this is a difficult task for most numbers. So we allow only the extensions that can be handled by rules that are already present.

Regrouping

Regrouping is changing the configuration of tokens on a board that preserves the sum of its local value. Now we describe the complete set of rules for regrouping. We call the set of rules complete if they enable changing a configuration of tokens into any other configuration that has the same value.

As shown in Figure 5, there are two rules of regrouping tokens on one rod:

(1) A token can be exchanged with two tokens that have the same value on the location just below.

(2) Two tokens of opposite values (1 and -1) can be put on, or removed from, any locations.

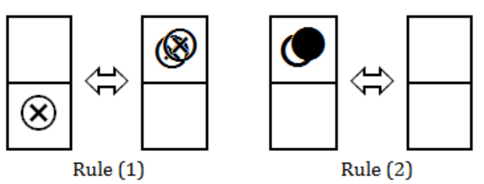


Figure 5. *Regrouping tokens* **(***X in a circle indicates a token that is either white or red).*

We define a one-to-one matching between locations of two rods by matching their roots and extending this matching up and down. (This may require adding more locations to a rod as described above.) We say that configurations on two rods correspond to each other if their lowest tokens occupy the matching locations.

When the root of one rod is a factor of the root of another rod, that is, r' = rf, where f is a whole number, then one token on the second rod can be exchanged for the corresponding configuration of tokens representing f in binary notation on the first rod (see Figure 6). The values 5*r*T, 2.5*r*T and 10*r*T are the same on each pair of rods.

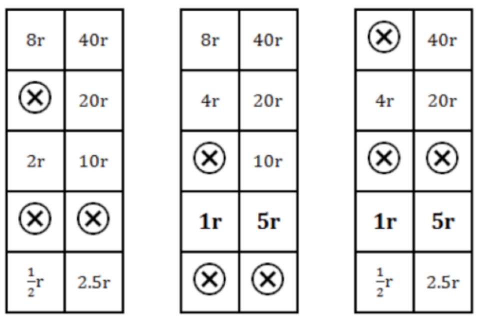


Figure 6. *Examples of regrouping (X in a circle represents any token (either white or red), and r and 5r represent roots. Here, the factor f = 5).*

Theorem. A composition of the rules (1) and (2), and rule (3) restricted to prime numbers f, is a complete system of regrouping rules, which is also independent, in the sense that no rule can be derived from the others.

Remarks

The restrictions that are put on adding new rods to a board guarantee that such an extension doesn't require one to adopt any new regrouping rules. So the set of numbers that are represented on one board can be described as a ring of integers in which only finitely many primes have reciprocals.

Using the Board in Different Grades

Boards for Beginners (grades K through 2nd)

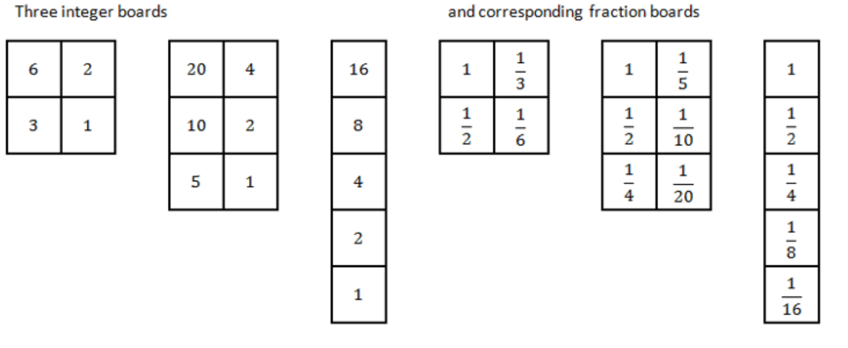


Figure 7. *Three integer boards and three corresponding fraction boards.*

Tokens are two-sided, with colors white and red. White has the value 1, and red the value -1. In general, one can put more than one token in each location. But the *capacity* of an integer board is the biggest integer that is represented with at most one token per location. So the capacities of the integer boards shown in Figure 7 are 12, 42, and 31. Most practical computations on a board are limited to numbers that are at most the double of the board's capacity.

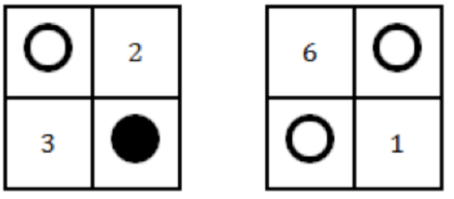


Figure 8. *The number 5 is on both boards.*

Using Negative Numbers

Positive and negative integers and fractions can be introduced at the same time because they are governed by the same set of rules. Using fractions is optional, because you may use just boards that have whole numbers as roots, and never use any locations below the root of a rod.

On the other hand, we think that red tokens need to be used, even when negative numbers are not discussed at the same time, as shown on the left in Figure 8. The algorithm for "taking away" positive numbers is more difficult, and harder to use, than the algorithm for adding negative numbers in which "white and red tokens cancel each other".

One can introduce red tokens without referring to negative numbers as follows:

Putting a red token on a board is a request to take out the white token from that particular location. During regrouping, when a white token lands on a red token, they are both taken away and the request is fulfilled.

A similar approach was used in China during the time when computation was done with rods. Red rods were used to represent positive numbers, and black rods were used for subtraction, without being treated as different numbers, but only as numbers to be subtracted (Hart, 2011).

Using Boards in Middle Grades (3rd through 8th)

Three main topics of elementary arithmetic are covered in middle grades: exact computation with (finite) decimals, exact computation with common fractions, and approximate computation with (finite) decimals. We do not treat the arithmetic of negative numbers as a separate topic, because we have already stated that negative numbers could be used as a computational tool, even before they are introduced as a new mathematical concept.

Decimal Boards

There are two main types of decimal boards: base 10, when powers of 10 are in the same row as 1; and base 5, when powers of 5 are aligned with 1 (as shown in Figure 9). Usually the length of each rod is specified, but the number of rods depends on the specific task.

...	...	...	...	...
...	20	2	.2	...
...	10	1	.1	...
...	5	.5	.05	...
...	...	...	...	...

...	...	...	...	...
...	10	2	.4	...
...	5	1	.2	...
...	2.5	.5	.1	...
...	...	...	...	...

Figure 9. *Two decimal boards; one is base 10 and the other is base 5.*

The rods on base 10 and base 5 boards are the same, but they are aligned differently. Base 10 boards are more useful for describing the standard algorithms that are taught in school because they mimic written computation in base 10. But base 5 boards are better for discussing more general aspects of processes of computation.

Multiplication and Learning "Number Facts"

Written algorithms require that students memorize appropriate "arithmetic facts". But the same algorithms, when executed on counting boards, do not require "facts", because the same results are achieved by applying regrouping rules.

We don't suggest that memorizing "facts" should not be required. We say only that they are not needed for executing algorithms on decimal boards. Actually in one study, 93% of college students, asked about their opinions, said that all school children learning mathematics should be required to memorize addition and multiplication facts (Baggett & Ehrenfeucht, 2011).
Here we show, using one example, a method of multiplication invented by John Napier (Gardner, 1986) that corresponds to modern multiplication in base 2, and that can be executed on all kinds of boards.

Explanation

On the base 5 board, shifting all tokens one space to the left multiplies by 5. And on any board, shifting each token one space up multiplies by 2. Shifting all tokens in several directions and adding the results can carry out any multiplication (See the example in Figure 10).

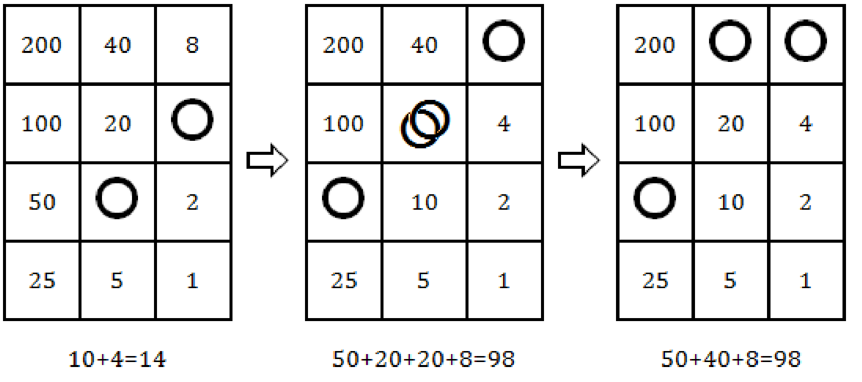


Figure 10. *Multiplying 14 by 7 on a base 5 board.*

Comment

We see that sliding the multiplier along the rod of the multiplicand, and adding its copies as you go along, carries out multiplication. It is essential that values of locations on a rod form a geometric progression. Geometric progressions were essential in the design of logarithms. And sliding one rod along the other was the principle underlying the invention of the slide rule.

Approximate Computation and the Simplest Representations of Numbers

Numbers that are represented on decimal boards are finite decimals only. They are the smallest set of rational numbers closed under addition and multiplication, which contains all integers and exactly two reciprocals of the prime numbers $2^{-1} = .5$ and $5^{-1} = .2$. This is really a "very small" subset of the rational numbers, so how the other numbers are approximated deserves special attention.

In school mathematics decimals include finite decimals and also infinite periodic decimals, which describe all rational numbers. And in secondary grades all irrational numbers are introduced as non-periodic decimals. So the question of approximations is usually reduced to one rule of "rounding".

Non-Adjacent Forms (NAF)

NAF notation uses three digits, 0, 1 and -1, to write binary numbers, but it requires that two consecutive non-zero digits are separated by at least one zero (Wikipedia reference).

Every number written in base two has a unique NAF representation, so when we use both red and white tokens on the same rod, every number has a unique representation, with the restriction that no two adjacent locations are occupied. Non-adjacent forms also use the smallest possible number of tokens that are needed to represent a number on any given rod. This representation has many uses and is more accurate when numbers are rounded or truncated, and it can be used on all boards. Figure 11 shows the numbers 7 and 13 in binary non-adjacent forms.

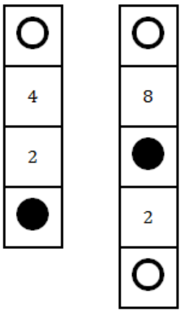


Figure 11. *Numbers 7 and 13 in binary non-adjacent forms.*

Some Limitations of Counting Boards

Each prime number that is represented on a board requires a separate rule. So using boards as a teaching tool becomes very limited when students start learning general properties of rational numbers. Specifically, the axiom that says, "Every rational number except 0 has a reciprocal" cannot be illustrated on any board.

Possible Changes in the Scope and Sequence of Teaching Arithmetic

The first, and the most radical, possible change is to start with positive and negative integers and binary fractions at the same time, and to use halving and doubling as the first arithmetic operations.

The second possible change is treating multiplication as a combination of shifting (doubling or multiplying by some specific primes) and adding, and not as repeated addition.

The third possible change is to use multiple bases and approximate computation before introducing all rational numbers (common fractions with arbitrary denominators). This change would provide the conceptual background for using all computer and calculator technology much earlier than is done now.

The fourth possible change is to introduce all real numbers, both rational and irrational, at the same time, after students learn arithmetic for numbers represented by finite decimals. It would provide a mathematical basis for many interesting applications. But it would shift the topic of common fractions and the distinction between periodic and non-periodic infinite decimals until the later grades.

Teaching Common Fractions Using Counting Boards

We give here an example of how to use counting boards to teach common fractions by using a specific board that handles fractions with denominators that are factors of 60, namely, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60. The operations that are allowed are addition and subtraction only. This method is still feasible when we take any number smaller than 1000 (instead of 60), but it starts being too complex for bigger numbers with many factors.

Addition and Subtraction of Fractions whose Denominators are Factors of 60

60	20	12	4
30	10	6	2
15	5	3	1

1	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{15}$
$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{10}$	$\frac{1}{30}$
$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{20}$	$\frac{1}{60}$

Figure 12. *An integer board and its corresponding fraction board with tokens to be added.*

On the fraction board we have a number, ½ - $^1/_3$ + ¼ - $^1/_5$, as shown on the right in Figure 13 To compute the total we put the same configuration of tokens on the integer board, as shown in the left in Figure 13.

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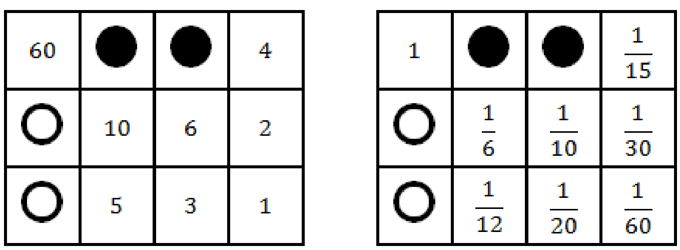

Figure 13. *The same configuration of tokens is put on the integer board. Now we regroup the tokens in Figure 13, getting 30 – 20 + 15 – 12 + 3 = 13, as shown in the left in Figure 14.*

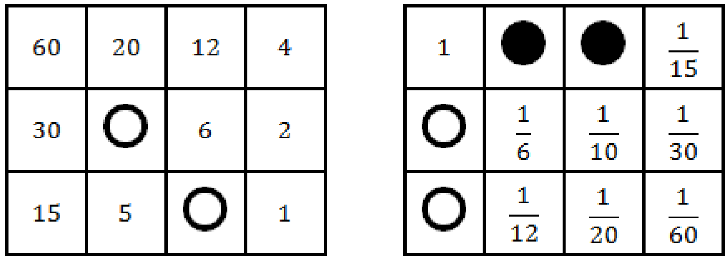

Figure 14. *On the integer board (on the left) the sum of 13 is shown.*

We copy this configuration on the fraction board, as shown in Figure 15:

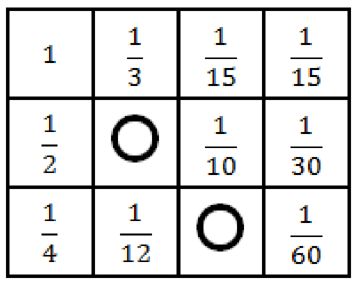

Figure 15. *The fraction board with tokens configured as they were on the integer board.*

This configuration on the fraction board (Figure 15) means that ½ - ${}^{1}/_{3}$ + ¼ - ${}^{1}/_{5}$ = 1/6 + 1/20 = 13/60. This method of computation is justified by the distributive law of multiplication over addition. The values on the integer board are the values on the fraction board multiplied by 60.

A bigger and more interesting example is the case with fractions with denominators that are factors of 840 = 8*3*5*7. Each board (integer and fraction) has four rows and 8 columns.

Testing the Use of Boards in Classrooms

The boards have been used for three different purposes:

- To teach standard algorithms for addition, subtraction, multiplication and division.
- To do tasks too difficult for students who fail to master written computations.
- To investigate properties of rational numbers.

Very few students (mainly pre-service teachers and elementary/middle school students) were interested in activities that involve standard algorithms. Students said that this topic was not interesting, because they did not need to learn more than one way of doing the same thing, and that doing the same thing in two different ways was confusing (Baggett & Ehrenfeucht, 2013).

Tasks involving computations were interesting when they were embedded in a "hands-on" activity, and not when they were given verbally. We were surprised that even college students preferred embedded tasks to those presented verbally (Baggett & Ehrenfeucht, 2017). Discovering properties of numbers that were represented on counting boards was the most popular topic. Such activities are not time-consuming; they are interesting to most students, and they often produce unexpected results. For example, we have presented the following problem to middle schoolers and preservice K-8 teachers: In how many ways can you represent the number 3 on a decimal board, using only two tokens? Remember that you may use negative numbers and decimals! (Some answers: 128 – 125, 2.5 + .5, 3.125 - .125).

Conclusions

At present, the methods of written computation are only a tool for teaching arithmetic. And the questions, what should be the scope and sequence of school arithmetic? What are the goals we want to achieve?, and What pedagogy should be used?, are currently among the central topics of mathematics education. In this paper we did not address any of these three questions. We described the construction and properties of counting boards designed on John Napier's principles that provide an alternative way of teaching arithmetic. We did not recommend that this new method should replace existing, well-tested teaching techniques. Instead we have shown by example how this new approach can be used within current classrooms as supplementary material to expand students' concepts about operations on numbers and properties of numbers. More such examples are in Baggett et al. (2017).

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