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The Effects of Mental Math Strategies on Pre-service Teachers' Self-awareness and Computational Skills

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The purpose of the study is to examine and proliferate pre-service teachers' understanding of mental addition and subtraction and prepare them for teaching these topics to their students. Thirty-one elementary pre-service teachers enrolled in two sections of an arithmetic course at a public university in the United States of America, participated in a semester long study. A mental math strategies program, lasting 10 minutes of each class period, was implemented with the treatment group consisting of seventeen pre-service teachers. The control group was taught the same material excluding the mental math strategies program. Both groups' strategies for solving one–step whole-number problems and the pre-service teachers' capacity to analyze their own strategies were examined. Results suggested pre-service teachers (both treatment and control) lacked knowledge for effective mental math strategies and were sometimes unaware of the strategies they used. This study presents novel approaches for improving pre-service teachers' mental math computational awareness and skills. The results of the study suggest investing time in teaching the ideas of mental math to pre-service teachers to help them attain fluency.

Keywords: Active strategies, arithmetic, mental math, pre-service teachers, variations.

With the *Common Core State Standards for Mathematics* (CCSSM) (National Governors Association for Best Practices, Council of Chief State School Officers [NGA & CCSSO], 2010) accepted in most U.S. states, strategies that ensure students' flexibility with their problem-solving approaches have become a necessary attribute of learning (Price, Mazzocco, & Ansari, 2013; Tsao, 2011). This increased role of mental computations is believed to promote conceptual understanding (McIntosh & Dole, 2000; Reys & Barger, 1994). Mental computation strategies require deep knowledge of mathematics (Heirdsfield & Cooper, 2002) and often provide early approximation to the correct answer (Plunkett, 1979). The current authors distinguish *active* mental math strategies from paper-and-pen algorithmic calculations even if they are performed mentally. The authors consider paper-and-pen algorithmic calculations a "passive strategy" that does not carry benefits of *active* mental math (Anghileri, 1999; Heirdsfield & Cooper, 2002; Plunkett, 1979).

Researchers (Carpenter, Franke, Jacobs, Fennema, & Empson, 1997; Fuson, Wearne, Hiebert, Murray, Human, Olivier, Carpenter, & Fennema, 1997; Klein & Beishuizen, 1998; Thompson, 1999a; 1999b; Van den Heuvel-Panhuizen, 2001; Yackel, 2001) subdivide mental math strategies into groups; the categories vary and do not always account for all possible methods. As a rule, these classifications reflect the authors' ideas behind particular studies and are not intended as guides. Such situations led to critiques by Threlfall (2002) who said, "general ways of making sense of mental calculation struggle to map onto the variations found in calculating particular problems" (p. 35) and concluded such a guide was impossible and unnecessary to make. Threlfall (2002) advocated for developing students' number sense instead of concentrating on teaching mental math strategies. The authors of this paper argue number sense and mental math strategies are connected. Using number sense is an integral part of applying mental math strategies (Heirdsfield & Cooper, 2002; Plunkett, 1979). Therefore, the ideas of fostering number sense by helping students construct active mental math strategies and then prompting them to apply contrasting active mental math strategies should not be ignored. Table 1 summarizes general strategies for active mental addition and subtraction: *separation*, *aggregation*, *compensation*, and *complementary addition*.

Table 1

Main Active Mental Math Strategies for Addition and Subtraction

Separation	1) 45 + 23 = 40 + 5 + 20 + 3 = (40 + 20) + (5 + 3) = 60 + 8 = 68
	2) 87 – 32 = (80 – 30) + (7 –2) = 50 + 5 = 55
Aggregation	3) 45 + 12 = 55 + 2 = 57
	4) 87 – 32 = 57 – 2 = 55
Compensation	5) 28 + 45 = 30 + 45 – 2 = 30 + 43 = 70 + 3 = 73
	6) 87 – 39 = 87 – 40 + 1 = 47 + 1 = 48
	7) 87 – 39 = 88 – 40 = 48
	8) 80 – 39 = 79 – 38 = 49 – 8 = 41
Mixed	9) 28 + 7 = 30 + 5 = 35 and 28 + 37 = 58 + 7 = 60 + 5 = 65
Aggregation or Compensation	10) 36 – 9 = 30 – 3 = 27 and 36 – 19 = 26 – 9 = 20 – 3 = 17
Complementary Addition	11) 51 – 49 = 2; Students count up solving equation: 49 + ? = 51

Separation strategy (Heirdsfield & Cooper, 2002) separates operands into parts, usually tens and ones (Yackel, 2001). *Aggregation* (Heirdsfield & Cooper, 2002) means starting calculations with one operand and adding or subtracting the highest place value part of the other operand, one number-part at a time (Yackel, 2001). *Compensation* simplifies calculations by changing one number while adjusting the other(s) (additive, subtrahend, minuend, or answer) to compensate for a change. In most of these cases, changes are introduced with the goal of rounding (see Table 1, examples 5–7, 9, and 10). The last entry in the *compensation* group illustrates an idea of adjusting the minuend. This idea is not cited as a popular strategy and is presented here to complete a general picture.

The current authors found some mathematical procedures (see Table 1, examples 9 and 10) belong to either the *aggregation* or *compensation* strategy, depending on the way one thinks about the calculations. Therefore, the *mixed* category was added to Table 1. The strategy 28 + 7 = 30 + 5 can be explained as changing 28 to 30 and adjusting 7 by reducing it to 5. On the other hand, the computation can also be explained as two-step addition: 28 is increased by 2 and then increased by 5. *Complementary addition* is the last strategy in Table 1. Although the strategy is quite beneficial, young students rarely use it (Fuson, 1982; Thompson, 1999b; 2000). This strategy is described by Carpenter and Moser (1979) as *Counting Up from Given*: "A child initiates a forward counting sequence beginning with the smaller given number **n**. The sequence ends with the larger given number **m**" (p. 39). Table 1 summarizes widely accepted active strategies and includes the *mixed* group and example 8 added by the authors. Following the ideas behind each approach, the current authors illustrate each strategy with examples written in a form of identical transformations to elucidate mathematics behind each approach.

There is more or less agreement between researchers regarding the types of mental strategies; however, there is no concord regarding the methods of teaching mental math. McIntosh, Nohda, Reys, and Reys (1995), described "at least three instructional approaches currently apparent in elementary classrooms" (p. 238). In constructivists and direct instruction approaches, students are encouraged or directly taught to develop active mental math strategies. In a third approach, "students are taught standard written methods for computing" (McIntosh et al., 1995, p. 239), and students are expected to compute mentally through this experience without any instruction on mental math strategies.

Unfortunately, for many decades, in most classrooms the third approach prevailed. McIntosh (1998) found U.S. elementary teachers spend about 50-90% of their time teaching pencil and paper algorithms. Current authors did not find any research indicating the situation with teaching mental math was somewhat different from 2005-2009 (approximately when the pre-service teachers, study participants, attended elementary school). The authors' experience with pre-service teachers from previous years allows them to hypothesize that mental math situations might have been the same until approximately 2009. Therefore, the authors' supposition is that pre-service teachers belong to a generation that was taught mostly algorithmic methods of computation and, as a result, have low proficiency with active mental math.

The current authors' hypothesis corroborated with studies of Şengül (2013) and Young-Loveridge, Bicknell, and Mills (2012) who found the number sense of pre-service teachers was very low internationally, particularly with problems involving fractions and decimals. The current authors speculated these deficiencies were rooted in insufficient mathematics skills on a much lower level. Yang (2007) recommended training programs should focus on developing computation and estimation skills of elementary school teachers who were responsible for developing children's number sense (Fung & Latulippe, 2010). Such training is necessary because mathematics teachers' obligation to the discipline increases when their mathematics-content knowledge is higher (Shultz & Herbst, 2016).

This study aims to improve pre-service teachers' knowledge regarding mental computations and to prepare them for teaching this topic to their future students. Hence the research questions are: how proficient are pre-service teachers at active mental math, and how can we improve their performance?

Theoretical Framework

In contrast with research (Heirdsfield, 2003) which states, students must not be taught active mental math strategies since they discover these strategies by themselves, many countries teach mental math strategies *via* class discussion or lecture and students are expected to explain the strategies they used verbally or in writing (Arginskaya, 1998; Peterson, 2006; *Primary Mathematics 2B Textbook*, 2003). To introduce the strategies, number lines, base-ten manipulatives, their pictorial representations, and/or a picture of 10 x 10-board are usually used in various countries (Arginskaya, 1998; Beishuizen, 1993). In the United States of America, a similar approach was recently proposed with a strong emphasis on whole-class discussion and concurrent usage of pictures, number lines, and Montessori 100-board. Application of this approach results in co-construction of mental math strategies by a "community of learners" (Heirdsfield & Lamb, 2006).

As discussed in Heirdsfield (2003), Vygotsky's zone of proximal development (ZPD) "was considered an important aspect of qualitative assessment of children's mental addition and subtraction proficiency" (p. 57). The current authors believe some students might have ZPD, which does not allow them to get all benefits when using models. Those students first must develop an understanding of mental math strategies using a *realistic*, hands-on approach. Base-10 manipulatives help develop necessary ZPD needed for using models.

To meet the needs of students with various ZPDs, the authors create an environment where students develop their own prerequisite experiences, individually discover new knowledge, and solve specially designed number problems to internalize this knowledge. To meet the first goal, the authors use base-ten manipulatives allowing each student to obtain individual, hands-on experience with addition and subtraction (see Figure 1) and create an environment to stimulate discovery. According to Montessori's (1982) ideas for *pink tower*, an environment with a reduced number of obstacles was prepared.

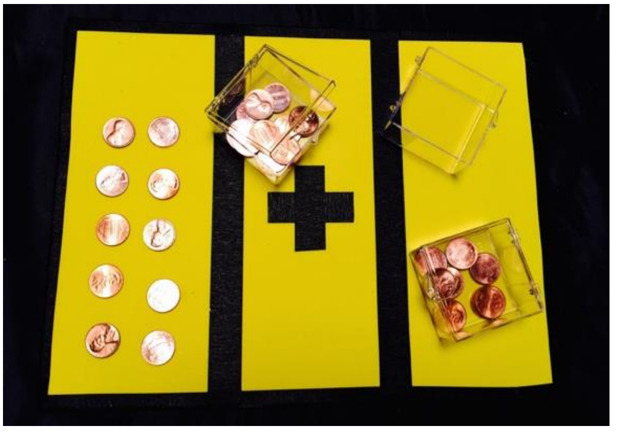


Figure 1. *(Left) Base-ten manipulatives.* **Figure 2.** *(Right) Pink tower.*

To make sense of different strategies and internalize the knowledge, students solved number problems which looked similar but required different hands-on manipulations and, consequently, different mental approaches. Such sets of problems were considered an analogy to a *pink tower* (Figure 2). Much like the cubes in this tower, made up of similar color and material, the posed problems looked very similar. As *pink tower* cubes had different sizes, the problems required application of contrasting mental math methods. Thus, for the problem, 28+5, it was beneficial to add 5 by breaking it into two parts, 28+5=30+3=33, whereas the problem, 23+5 did not require any "tricks" and could be solved in one step.

When students solved the mixture of two types of problems presented above, the idea of "addition by parts" was comprehended well due to contrasting problems. The variations in "methods of solving" for similar problems helped students better comprehend each method.

The ideas of proposing simple, contrasting-approach problems strongly correlate with Marton and Pang's theory (2006) of necessary conditions of learning. The prepared environment allows students to make sense of one objective at a time and offers contrasting variations; which, according to Marton (2015) provides necessary conditions of learning. Recent studies (Bofferding, Farmer, Aqazade, & Dickman, 2016; Pang, Marton, Bao, & Ki, 2016) demonstrate growing interest toward implementing variations into educational mathematical practice, including number-calculations.

Authors' Approach for Teaching Active Mental Math Strategies

To trigger discovery and help students internalize knowledge, the current authors implemented the following approach:

a) Students were taught one learning dimension (Marton & Pang, 2006) at a time (see examples in Table 2) while ideas were validated through hands-on experience.

b) Students used manipulatives and pictures to solve problems in a *prepared environment*—the best strategies became obvious to the students who used provided base-ten manipulatives.

c) Guided questions framed students toward discovery (Vygotsky, 1978; Wood, Bruner, & Ross, 1976).

d) Contrasting variations (Marton, 2015) were applied to stimulate learning.

e) Direct instruction (Magliaro, Lockee, & Burton, 2005) was used to facilitate translation of hands-on strategies into mathematical language.

Direct instruction was used only for translation and only self-discovery was used for creating novel procedures. Without such instruction, the base-10, hands-on approach would result in a "pseudo-effect." According to research by Beishuizen (1993), when students were not translating each step into mathematical language, despite their success with manipulatives, students did not translate hands-on mental math manipulations into mental computations.

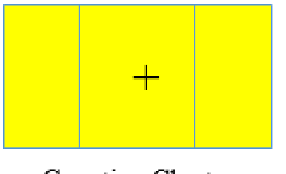
Methodology

Participants and Course Materials

Participants of this study were thirty-one elementary pre-service teachers from a western public university enrolled in two sections of an arithmetic four-credit course. The lead author was the instructor for the treatment group consisting of one section with 17 pre-service teachers. The remaining 14 participants were the control group. A different instructor taught these participants. The treatment group met two days per week for 2 hours a day. The control group met three days per week for 1 hour and 20 minutes a day.

The same standard course textbook was used for both groups with an additional mental math strategies program provided only to the treatment group. The second author developed the mental math strategies program (MSP) for remediation and enrichment of 2nd to 7th grade students for a learning center in the Midwest U.S. The lead author used 10 minutes from each class to work on MSP for addition and subtraction, utilizing base-ten manipulatives (see Figures 1, 3) for most of the lessons.

Materials: pennies or plastic coins, plastic boxes that can hold 10 coins, a colored rectangular sheet of paper (counting chart). The chart is subdivided into three equal parts using two parallel lines.



Counting Chart

Figure 3. Materials needed for activities.

Lessons included class discussions of the mental math strategies, students' presentations, hands-on addition/subtraction, and practice with step-by-step identical transformations used to mathematically describe strategies. During class-time, students worked individually and in groups using manipulatives to construct mental math strategies; afterwards, they translated obtained strategies into mathematical equations. Table 2 below outlines some of the approaches from the MSP.

MSP Approach

To illustrate how to implement the contrasting variations (Marton & Pang, 2006), the current authors presented a 4-Topic unit for teaching aggregation and separation tactics in addition (see Figure 4 for Topic 1). The following are the topics-- Topic 1: Magic Numbers Model and Separation Method; Topic 2: Developing Aggregation Strategy for Addition; Topic 3: Separation and Aggregation Methods- Contrasting; Topic 4: Intensified Training with Contrast. The authors first introduced a model (Topic 1) and provided a one-parameter environment for two addition strategies: separation (Topic 1) and aggregation (Topic 2). Then, the authors provided variations with a contrast—pre-service teachers must differentiate between addition by separation (Topic 1 problems) and addition by aggregation (Topic 2 problems) approaches. Finally, in Topic 4, the range of variations was extended.

Data Collection and Analysis

Four pre-tests, one mid-term test, and one post-test from the treatment group were collected; only one post-test from the control group was collected for data. Four timed pre-tests with basic arithmetic problems on whole numbers were administered to the treatment group during the first week of classes. The first pre-test (*pre-test 1*) was ten minutes long and consisted of 110 problems, most involving 2-3 digit numbers. The goal was to see how many problems participants could solve mentally. The second test (*pre-test 2*) was five minutes long and consisted of six addition/subtraction problems, as well as multiple-choice questions asking the participants to identify the strategies used in those problems. The third pre-test (*pre-test* 3) was 5-minute long and consisted of two 4-digit addition/ subtraction problems along with ten 2-digit addition and subtraction problems. The fourth test (*pre-test 4*) was 5 minutes long and consisted of eight 3-4-digit addition/ subtraction problems. No homework was assigned to the participants on this material for the first six weeks. Only 10 minutes from each class was used to work on MSP for addition and subtraction. After six weeks, pre-service teachers had a five-minute mid-term test consisting of 80 two-three-digit addition/ subtraction problems similar to the pre-test addition and subtraction problems. The participants were asked to use the mental math strategies learned in class, present identical transformations that reflect their mental computations, and provide feedback regarding mental math strategies used.

Table 2
Some of the Topics on Mental Math

	Lessons	Examples
1.	*Building numbers* using base-ten manipulatives.	Represent 26 as two closed boxes of ten and additional 6 coins.
2.	*Separation strategy of addition and subtraction using base-ten manipulatives.*	Perform addition, 23 + 45 using boxes of tens and coins. Perform subtraction, 45−23, using boxes of tens and coins.
3.	*Close the box* method for addition to understand *adding to ten.* (***Mixed***)	Add 38 + 7 by manually adding coins into the box with 7 coins. Writing 38 + 7= 40 + 5= 45.
4.	*Separation of ten* method. Use base-ten manipulatives to develop strategy for solving 40 – 6 and 150 – 7 problems.	Translate manual actions onto symbolic language for the problems like $40 - 6 = 30 + 4 = 34$. Extend experience to solve more complex problems: $150 - 7 = 140 + 3 = 143$.
5.	*Empty the box* method for mental subtraction. (***Mixed***)	Subtract, 35−7 using manipulatives. Present manual steps using identical transformations: $35 - 7 = 30 - 2 = 28$.
6.	***Complementary Addition*** method for subtraction of *big* numbers.	51− 49 = ? presented as 49 + …= 51. Here, students count from 49 to 51 and write what is needed to reach 51.

Note. The grey row is discussed in detail in *Figure* 4.

After the mid-term test, daily homework, 100–120 multi-digit addition/subtraction problems, was assigned weekly for four weeks. A post-test was administered to both the control and treatment groups. It consisted of a 3-minute timed test of 3 and 4-digit addition/subtraction problems (20 problems) and a timed 5-minute strategies' explanation. During the 3-minute test, pre-service teachers were asked to write their answers. During the 5-minute task, the participants explained their strategies for six multi-digit addition-subtraction problems.

All written data was digitized. A spreadsheet of the pre-service teachers' written responses to each task was created to trace an individual pre-service teacher's progress across tasks and simultaneously compare their responses over time. The experimental data were analyzed quantitatively.

Results

Pre-service Teachers' Mental Math Flexibility

Four timed pre-tests, conducted with treatment group, captured pre-service teachers' mental math strategies. Below, the results of four pre-tests, which examined pre-service teachers' readiness in terms of active mental addition and subtraction approaches, are discussed.

First, data from *pre-test 1* revealed pre-service teachers used time-consuming techniques, e.g., standard algorithms for addition and subtraction in simple 2–3 digit number problems, which could have been solved faster using alternative strategies. *Pre-test 2* demonstrated some pre-service teachers identified their work to be based on mental math strategies but actually used standard algorithms (Figure 5).

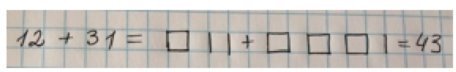
Topic 1. Magic Numbers' Model and Separation Method.

Objective: ***i.*** *introducing the basis for using the Magic Number model for further lessons.* ***ii.*** *using a one-parameter environment to trigger discovery of "addition by separation" method.*

Magic Numbers' Model: Magic numbers are numbers presented by using closed boxes, each with 10 coins and some "free" coins aside or pictorially. The figure on the left shows the Boxes-coins representation: *On the left in the counting chart, a closed box and 8 coins represent 18. On the right, a closed box represents 10 and an open box represents 5.*

<u>Rules for creating *Magic Numbers*:</u> (i) fill boxes with 10 coins; (ii) a box considered filled and must be closed as soon as it has 10 coins; (iii) do not close a box if it has less than 10 coins; (iv) there are no boxes with more than 10 coins.

The figure on the right shows a pictorial representation: *squares represent tens; "sticks" represent "free coins." The addition of 12 and 31 is presented in the example.*



Topic 1 problems' specifics: *All numbers are less than 100. One-digit number is added to a two-digit number. Addition of ones results in 10 or lower answer.*

Figure 4. *Topic 1— The magic numbers' model and separation method.*

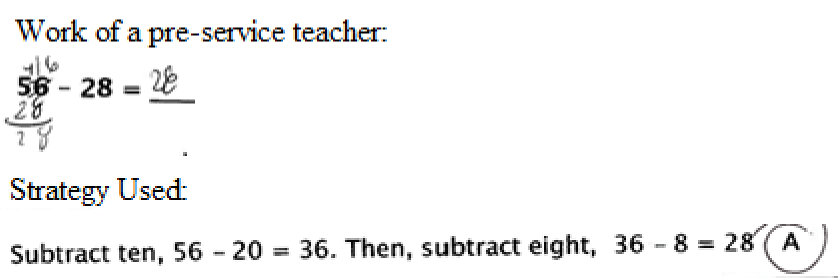


Figure 5: *Work and strategy chosen by a pre-service teacher.*

The current authors argue the inconsistency between calculations and the chosen strategies did not depend on the multiple-choice test layout or inattentiveness. Most of the answers were not chosen arbitrarily; on similar problems, the participants consistently chose similar strategies. Therefore, it was concluded, the discrepancies between the methods applied and the strategies selected were due to pre-service teachers' insufficient knowledge regarding the methods used.

Many of the students performed two-digit addition/subtraction mentally on *pre-test 3*, but could not mentally solve problems involving multi-digit problems that could have been easily solved using active mental math strategies. Fifteen out of 17 pre-service teachers used only standard algorithms for 4-digit subtraction and addition (Figure 6) while the problems could have been solved effectively using compensation, $4612 - 2997= 4612-3000 + 3= 1612 + 3= 1615$; $5894 + 1998= 5894 + 2000 - 2= 7894 - 2= 7892$.

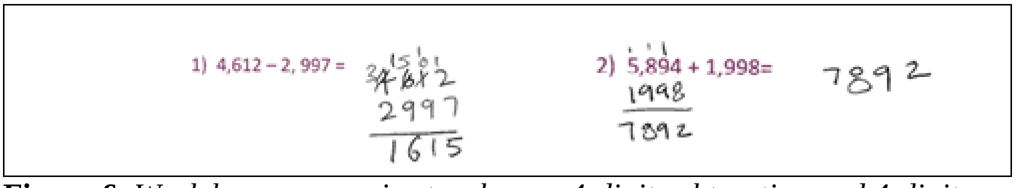


Figure 6. *Work by a pre-service teacher on 4-digit subtraction and 4-digit addition.*

On *pre-test 4*, only 13 out of 17 students were present in-class. Only one student used a mental math strategy—separation method, to solve most of the problems, while seven out of 13 students used the algorithmic method for all problems but two. Those two exceptional problems, 692-689 and 471-467, required complementary addition. The rest of the students, 5(13), used algorithmic methods for all problems. At least 8 of 17 pre-service teachers demonstrated competency with complementary addition, 5 did not use complementary addition, and 4 were absent in the day of *pre-test 4*.

Intermediate Progress after Mid-semester Treatment

Analyzing the mid-term data from the treatment group, it was observed, the pre-service teachers continued to struggle with problems like $60 - 8$ and $150 - 7$. They could not solve these problems fluently even after the lessons where they *constructed* the strategy of "separation of 10" and had some practice with this method.

Mid-term data also suggested a prepared environment and scaffolding helped most of the pre-service teachers construct the efficient methods for addition and subtraction. It included separation, aggregation, and compensation strategies (Figure 7). However, the pre-service teachers needed additional practice with novel strategies to become accustomed to the ideas of mental math.

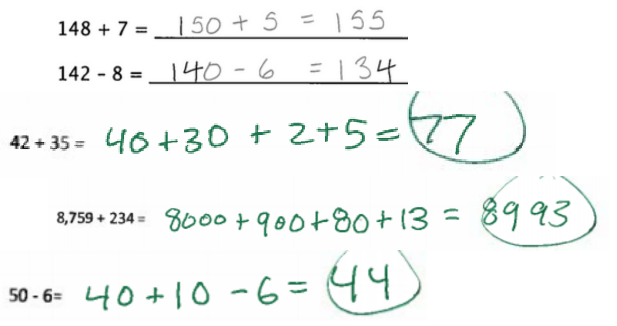


Figure 7. *Pre-service teachers' work on mental math strategies.*

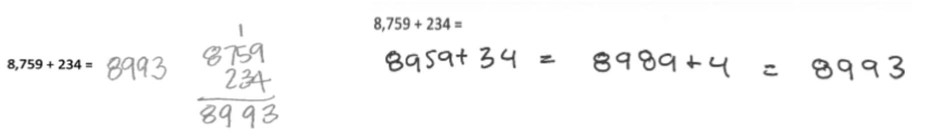
Providing feedback in the mid-term test, one pre-service teacher wrote, "Because, they [the methods of calculations] are different than what is always done, so it's a new concept to me." Another wrote "I understand it while we are doing it, but then because I haven't practiced, I don't remember." The average correct response in the mid-term test (5-minute mental math test) was 51 problems out of 80. Unfortunately, the current authors could not compare the results with the *pre-test 1*, since the *pre-test 1* contained all four arithmetic operations. At the beginning of the study, the researchers wanted to teach calculations involving 3-4 digit positive numbers and include topics on multiplication and division. However, they found, pre-service teachers, who participated in the study, were not fluent with active mental math strategies for addition and subtraction of two-digit numbers.

As it was discovered on the mid-term test, the pre-service teachers' awareness toward mental math increased significantly; 16 of 17 pre-service teachers expressed interest in learning mental math strategies. The study participants started to analyze their calculations and recognized better alternatives to their strategies. Meanwhile, 10 of 17 pre-service teachers admitted they needed more practice to achieve fluency.

Pre-service Teachers' Progress after Treatment

In the post-test, pre-service teachers showed improvement in the use of mental math strategies. Thirteen pre-service teachers from the treatment group used active mental math strategies while all 16 students from the control group used a passive strategy—standard algorithms (Figure 8). In spite of this, the treatment group's results were somewhat lower than the control group on the timed post-test with 3-4 digit addition and subtraction problems. On average, the treatment group students solved 8.4 out of 20 problems correctly while the control group solved 8.7 problems correctly.

Table 3 demonstrates pre-service teachers' (treatment group) use of mental math strategies after instruction. The table demonstrates that after the treatment, fifteen out of 17 used mental math strategies and two used only algorithmic calculations. There is not enough evidence to say, in all cases, students applied the best strategy in terms of working memory use.



Control Group Treatment Group

Figure 8. *Strategies of pre-service teachers from treatment and control groups on the timed post-test.*

Table 3

Awareness and Flexibility with Active Mental Math Techniques

Mental Math Strategies	**12,789-3,123**	**4,321+1,908**	**509-286**	**8,759+234**	**42+35**	**50-6**
Separation			1	1	3	11
Aggregation	13	13	8	11	11	
Compensation	1	1	4	1		
Mixed	1	1	2			
Other Non-Algorithmic				2		1
Total number of Non-Algorithmic	15(17)	15(17)	15(17)	15(17)	14(17)	12(17)
Algorithmic	2(17)	2(17)	2(17)	2(17)	1(17)	
Unclear					2(17)	5(17)

Note. Complementary addition was not part of the post-test because just after one lesson; all 17 students in class showed fluency on typical complementary addition problems, for example 32-29 or 23-19.

Discussion and Conclusions

First, this study examined pre-service teachers' readiness in terms of active mental addition and subtraction and found their preparedness to be low. Pre-tests and mid-term test showed pre-service teachers' deficiencies with number sense might originate from their low proficiency with active mental addition and subtraction. Applying a series of pre-tests, this study found, pre-service teachers, contrary to children (Fuson, 1982; Thompson, 1999b; 2000), demonstrated better competency with complementary addition than with other active mental math strategies. After acquiring algebraic ideas in higher grades, the current authors hypothesized, the study participants developed a deeper understanding of connections between addition and subtraction and, as a result, attained fluency with complementary addition. This supposition led to a conclusion that complementary addition required, besides number sense, an *algebraic sense*.

It must be added, besides insufficient fluency with mental computations, some discrepancies between the methods the pre-service teachers thought they applied and the strategies they utilized were found. Based on this, the current authors argued, the reason behind low proficiency and low flexibility with mental techniques was an insufficient metacognitive awareness regarding the variety of mental math approaches. Hence, the authors concluded, pre-service teachers must be helped in constructing active mental math strategies and attaining flexibility and fluency when using them.

On a small sample of 31 students, it was found, with an exception of complementary addition strategy, most teacher candidates did not construct their own mental math strategies throughout their school years but have shown high proficiency with algorithmic calculations. Therefore, if it is expected from elementary teachers to come to schools and help children develop flexibility with mental math strategies, teachers' educators must assure the teachers are aware of these active mental math strategies. The current authors believe the poor situation with mental math can be improved using supplementary arithmetic instruction based on an application of Martin and Pang's contrasting variations (2006). In accordance with this theory, students are placed in an environment prompting them to "construct" each strategy and internalize it by solving problems using contrasting strategies (contrasting variations).

In an ideal case, students would construct new strategies themselves or with very little help. However, most pre-service teachers were not new to the idea of regrouping. Therefore, in many cases, the teacher candidates enforced the knowledge of familiar concepts instead of constructing new knowledge. Despite this limitation, a huge leap in the pre-service teachers' mental math appreciation, as well as with their engagement with the strategies was detected after only a few weeks of 10 minutes-per-lesson instructions. This progress demonstrated, direct instruction with translating "hands-on" strategies into mathematical equations helped pre-service teachers avoid the "pseudo-effect" described by Beishuizen (1993) and transformed their "hands-on" manipulations into analogous mathematical calculations.

Nevertheless, as the mid-term test demonstrated, constructing new knowledge followed by 5-minute in-class practice is insufficient to change students' habits of calculation using passive algorithmic strategies. The current authors' experience (unpublished results) shows that all students, who have sufficient practice with variations, develop complete proficiency with active mental math strategies. Heirdsfield (2003) found, young students did not need to be taught active mental math strategies but "merely encouraged to develop and use efficient strategies" (p. 60). However, current research with adult students demonstrated otherwise. With minor exemptions, study participants customarily used the *algorithmic* approach and did not develop effective approaches for arithmetic calculations. The mid-term test showed that an encouragement and help with "constructing" more productive (in terms of working memory usage) approaches resulted in minor progress with strategies application.

Therefore, after the mid-term, the authors adjusted their approach by adding additional pen-and-pencil practice (100-120 problems weekly) with mental math strategies. Pre-service teachers used identical transformations to solve and present step-by-step solutions for number problems while applying contrasting mental math strategies.

The combination of 10-minute class discussions and/or pop-quizzes with homework significantly improved pre-service teachers' flexibility with mental math approaches. Despite this, study participants did not reach mastery levels comparable to their mastery with algorithmic procedures. This was seen from a comparison of the post-test written by the control group (algorithmic calculations) and the treatment group (active mental math strategy based calculations). Post-test demonstrated, extensive guided practice with applying strategies flexibly is needed if we want our teachers to serve as role models for their future students.

For this research, a small sample was chosen to closely monitor individual response to a treatment, adjust treatment if needed, and provide individualized instruction to pre-service teachers. In the future, the authors plan to incorporate necessary adjustments and examine a larger sample of participants. In addition, further work is needed to better evaluate progress after implementation of the mental math supplementary course. Particularly, it is necessary to design better pre-tests on addition and subtraction, supplement the tests by interviewing students, and videotape their work to better evaluate the strategies students used. Additional practice in the form of homework assignments from day one of the supplementary course would further improve the quality of learning. The last supposition is supported by (a) pre-service teachers' remarks regarding necessity of more practice and (b) by the current authors' experience with young students (non-published results), who rapidly developed flexibility with taught strategies after intensive practice with sets involving contrasting-strategies problems.

The results show, the elementary pre-service teacher participants did not possess the necessary skills for teaching mental math strategies to 2nd and 3rd graders and needed extensive training to increase their self-awareness and develop these skills. The proposed 10 minutes per class supplementary mental math instruction increases the pre-service teachers' awareness regarding active mental math strategies and is recommended for use in teacher-preparation program. However, increasing pre-service teachers' active mental math skills to the level where it can compete with algorithmic mental calculations requires more extensive efforts.

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