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Fostering Novice Teachers' Knowledge of Students' Errors on Fraction Division by Using Researched-Based Cases

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There are increasing studies on knowledge of students' learning, but limited study focuses on detecting students' errors gathered in authentic classrooms and repairing them. Novice primary teachers, as well as students, have difficulties with dividing fractions and have weak knowledge of students' understanding and misunderstanding. Thus, this study was designed to foster novice primary teachers' knowledge of anticipating, identifying, and correcting students' errors on dividing fractions through using research-based cases. The data were collected from a case-based discussion group including four novice teachers and a facilitator. Five case-based discussions were tape- and video-recorded. A pre- and post-test consisting of four items with 16 sub-items were carried out. The result shows that the novice teachers became more versatile in anticipating students' errors and shifted from algorithms based errors toward conceptually based errors. The pedagogical strategies of repairing students' errors to be used included to help students revisit their prior knowledge, understand semantics structure of fraction division word problems, and realize the nature of fraction division. They were able to offer various pedagogical strategies corresponding to specific errors to repair students' errors.

***Keywords*:** Research-based, novice teachers, teacher's knowledge of students' errors, fraction division.

Fraction is considered as a complex mathematical content in elementary school mathematics (Mack, 1990). The complexity results into more difficulty on fractions than other topics for students and teachers (Rizvi & Lawson, 2007; Tirosh, 2000). Particularly, pre-service and novice primary teachers have difficulties with dividing fractions (Tirosh, 2000). Research suggests that novice teachers lack of deep understanding of fraction division and have weak knowledge of students' misunderstanding (Young & Zientek, 2011)

Traditionally, teachers hide and ignore students' mathematical mistakes and make students afraid to make mistakes (Durkin & Rittle-Johnson, 2015). If a teacher ignores students' errors (SE) or does not correct student's error, the error may be repeated or accumulated or cause other related errors. Nevertheless, the detection of errors is a pre-condition for revising false knowledge, and therefore preventing the errors from reoccurring. Research shows that some of in-service teachers are short of knowledge to identify SE and the causes of the errors of dividing fractions (Tirosh, 2000). Novice teachers with little teaching experience have weaker knowledge of students' errors and pedagogically repair the errors of fractions (e.g., Lo & Luo, 2012). Thus, novice teachers' (NTs) about students' learning fraction division should be enhanced.

Fennema, Carpenter, Franke, Levi, Jacobs, and Empson (1996) suggest that improving knowledge of students' misconceptions can probably only be acquired in the context of teaching. This implies that teachers should learn about students' difficulties via classroom teaching. However, it is difficult for NTs to use SE as a teaching tool without similar learning experience in teacher preparation. The use of cases appears to be a way of fostering NTs to develop this knowledge, since the cases include various errors of a specific content collected from authentic errors.

This study focuses on teachers' identification of students' errors gathered in realistic classrooms. It aims to examine the effect of the research-based cases on helping novice primary teachers to identify the students' errors, their causes, and figure out possible strategies to repair the errors. Two research questions consisted of: (1) What are the students' errors of fraction division the novice teachers identified before and after the case discussion workshops? (2) What are the pedagogical repairs of students' errors of fraction division the novice teachers suggested before and after the case discussion workshops?

Theoretical Background

The Importance of Knowledge of Students' Errors

There are several aspects of knowledge of students' errors or misconceptions, such as anticipating and diagnosing students' errors. Diagnosing errors is important for understanding students' changing knowledge (Durkin & Rittle-Johnson, 2015). The importance of knowledge of students' errors (KSE) can be illustrated by Minsky's (1994) theory of two complementary types of knowledge: negative and positive knowledge. Negative knowledge means knowing about incorrect concepts and procedures. We are not usually taught incorrect concepts or procedures but we sometimes have errors or misconceptions. The Chinese slogan "failure is a mother of success" means the use of errors can be achieved into success, since errors can be used as a resource of detecting the causes of error or as resources for promoting learning (Borasi, 1994; Heinze & Reiss, 2007). Misconceptions or errors can persist over a long time and must be overcome (Eryilmaz, 2002). Hence, one can learn from handling errors. Teaching to hide misconceptions could result in misconceptions, which might be hidden from the teacher and from the children themselves. In contrary, if there is clear feedback or dealing with errors during the learning process, it seems to be more effective than hiding it (Keith & Frese, 2008). If a teacher is familiar with various causes of errors in children's responses, she/he would be likely to select an appropriate instructional strategy. If a teacher makes an appropriate use of students' errors by searching for the causes of errors and using different strategies of handling errors, students have a chance to improve their conceptual understanding (Borasi, 1994; Heinze & Reiss, 2007). Hence, teachers need to enhance their knowledge of handling SE. Given this, identifying errors, detecting their causes, and handling the errors are three aspects of teachers' knowledge of SE referred to the study.

Research on Knowledge of Students' Learning of Fraction division

Lo and Luo (2012) summarize five structures of word problems of fraction division from previous studies (Ma, 1999; Reys, Lindquist, Lambdin, & Smith, 2012; Van de Walle, 2007). The five structures consist of: (1) equal-group measurement division, (2) equal-group partition division, (3) comparison measurement division, (4) comparison partition division, and (5) rectangular area division. Division problems embedded in the first two structures deal with a certain number of groups, all of equal size. Comparison problems deal with multiplicative comparison situations; one set involves multiple copies of the other. In rectangular area problems, the cross-product consists of a two-dimensional unit, such as length x width = area of a rectangle (Lo & Luo, 2012; Ma, 1999).

Fraction, involving complex understanding, challenges students to learn because it defies students' intuitions from whole numbers. Tirosh (2000) summarizes the literature on SE on fraction division of fractions and identified three main categories; (1) algorithmically based mistakes, (2) intuitively based mistakes, and (3) formally based mistakes. Various bugs in computing division expressions are included in the algorithm-based mistake. The intuitively based errors result from intuitions held about division. According to Tirosh, students have a tendency to overgeneralize properties of operations from natural numbers to fractions and to interpret division primarily using a partitive model of division. However, the primitive model of division imposes three constraints on the operation of division: (a) the divisor must be a whole number; (b) the divisor must be less than the dividend; and (c) the quotient must be less than the dividend.

Research on knowledge of school mathematics has consistently reported on teachers' difficulties with understanding fractions (Isiksal & Cakiroglu, 2008; Son & Crespo, 2009; Tirosh, 2000). For instance, the primitive model has been shown to seriously limiting children's and prospective teachers' abilities to correctly respond to division word problems involving fractions (Tirosh, 2000). Research shows that the pre-service teachers are skilled in solving the division of fractions, but their levels of reasoning for explaining the meaning of the operation are low (Isik & Kar, 2012; Son & Crespo, 2009). Borko and colleagues' study (1992) reveals that prospective teachers majoring in mathematics are skilled in "inverting and multiplying" algorithm when dividing fractions, yet fails to explain it with conceptual understanding to sixth graders. The result is consistent with Li and Kulm's study (2008) in which many teachers in US. are unable to explain why the division algorithm "flip and multiply" works. In addition, Mack (1990) states that prospective teachers are unaware of the major sources of students' incorrect responses to fraction division.

The Uses of Cases

Cases can provide users with an opportunity to focus on students' thinking and show the explanations regarding what students appear to know and understand the mathematics they are learning. Case discussion plays an essential role in the use of cases. Discussing cases creates the possibility of cognitive conflicts from others' that leads to adjust one's own thinking (Lin, 2010). Through the process, users expand what students are able to do, and get more familiar with students' misconceptions that are common in a particular domain. Thus, cases can help users to develop knowledge of students' learning (Stein et al., 2009). Cases differ according to grade level, content focus and type. Cases in mathematics teacher education share a common feature of providing authentic context for helping teachers develop skills of analysis and gain broad repertories of pedagogical technique (Lin, 2005; Merseth, 1999).

Lin (2005) describes two features of research-based cases: (1) the cases were constructed by classroom teachers and the researcher involving in when the cases were constructed. (2) the instructors in the written cases were involved in the research context. Research-based cases are real and based on valid research. The research-based cases include students' various correct and incorrect solutions gathered in authentic classrooms. They provide teachers with opportunities to identify where the errors occurred and to anticipate SE or difficulties. This study takes the research-based cases (RBC) discussion as the material for discussion in the workshops for enhancing NTs' knowledge of students' difficulties and mistakes (Lin, 2005; 2010).

Research Method

Participants

Four novice primary teachers with less than three years of teaching and a facilitator (T0) set up a cases-based discussion group. Their academic backgrounds were not related to mathematics education, but they were eager to improve their mathematics teaching. Their participation in the study was based on their willingness to improve their expertise in mathematics instruction. The facilitator with eight years of teaching experience played the role of the leader in the research-based case (RBC) small group discussion. The facilitator has received thirty-six 3-hours workshops of cases-based discussion which were held by the researcher.

The Research-Based Cases Workshops

Six 3-hours RBC workshops related to fraction division were held within a semester. Five written cases from a casebook were discussed in the six RBC workshops. Each written case consists of seven sections: Introduction of the case, Grade level, Prior knowledge, Objectives of the case, Context of the case including problems that were posed, Dialogues between teacher and students, Students' authentic correct or incorrect solutions, Discussion questions, and User's guides.

Case 1 demonstrates students' strategy "dividing the numerators and denominators" for solving the type of $\frac{b}{a} \div c$ in the context of partition division. For example, *A cup of Coke with 6/5 liters is equally shared with 4 friends. How much of the Coke will each friend get?* Case 2 describes students' strategies for solving the type of ($\frac{b}{a} \div \frac{d}{c}$, c a factor of a) in the context of comparison measurement division (Lo & Luo, 2012). Case 3 displays students' nine solutions of solving the problem $\left(a \div \frac{c}{b} \right)$ without a remainder in the context of measurement division. Case 4 displays students' nine solutions of solving the problem $\left(a \div \frac{c}{b} \right)$ with a remainder under the context of measurement division. For example, *A bucket of paint contains 5 liters. Every 2/3 of liters is poured into a bottle. How many bottles of the paint are needed? What is the remainder of the paint?* Case 5 describes students' strategies for solving the type of ($\frac{b}{a} \div \frac{d}{c}$) in the context of comparison of partition division and the algorithm of inverting and multiplying for the fraction division. For example, *a bag of rice weighs 12/16kg. A bag of beans weighs 1/4kg. How much does the bag of rice weigh as compared to the bag of beans?*

Data Collection

Prior to each workshop, all participants were required to read an assigned written case in advance. Each RBC workshop started with one-hour small group discussion. During discussion, the participants were also commonly asked to answer the questions: (1) What is the main mathematical idea in the case? (2) Which of the solutions are incorrect? (3) What might be the causes of the errors listed in the case? (4) What pedagogical strategies you would like to respond to each error? The workshops were video recorded.

The participants were administered pre-test and post-test on SE in fraction division. Two dimensions were involved in the framework of developing the items (see Table 1). One dimension was knowledge of SE of dividing fractions, consisting of anticipating students' possible errors (AE), detecting the errors given in solutions (DE), and handling the errors (RE). The other dimension was the semantic structure of fraction division, consisting of equal-group partition division (EPd), equal-group measurement division

(EMd), comparison of measurement division (CMd), and comparison of partition division (CPd). We also consider the remainder is nonzero.

Table 1

Framework of Pre- and Post-Test

Knowledge of students' errors / Semantic structures	AE	DE	CE	RE
Equal-group partition division(EPd) (*A cup of milk with 8/5 liters is equally shared with 3 friends. How much of the milk will each friend get?*)	Item 1(1)	Item 1(2)	Item 1(3)	Item 1(4)
Equal-group measurement division (EMd) (*A bucket of apple juice contains 7 liters. Every 2/5 liters is poured into a bottle. How many bottles of the juice are needed? What is the remainder of the apple juice?*)	Item 2(1)	Item 2(2)	Item 2(3)	Item 2(4)
Comparison partition division (CPd) (*A bag of green beans weighs 16/9kg. A bag of red beans weighs 1/3kg. How much does the bag of green beans weigh as compared to the bag of red beans?*)	Item 3(1)	Item 3(2)	Item 3(3)	Item 3(4)
Comparison measurement division (CMd) (*A bottle of water is 8/5 liters; the water in a bottle is 3/4 times more than in a bucket. How much water is in a bucket?*)	Item 4(1)	Item 4(2)	Item 4(3)	Item 4(4)

AE: Anticipating possible errors DE: Detecting errors CE: Interpreting causes RE: Repairing errors

The pre-test and post-test were administrated in two stages. Post-test apart five months from pre-test with the same items were conducted.

Stage I is to anticipate students' possible errors by asking "*what are possible errors or mistakes students may make when solving the problem?*" An item 2(1) with the context of EMd is shown as follows.

"*A bucket of apple juice contains 7 liters. Every 2/5 of liters is poured into a bottle. How many bottles of the juice are needed? What is the remainder of the apple juice?*

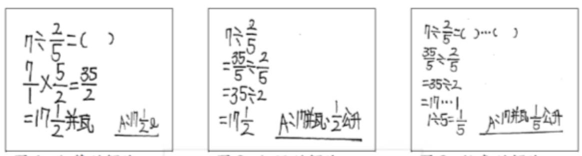
(1) what are possible errors or mistakes students may make when solving the problem?"

Stage II is to explore teachers' knowledge of detecting students' errors and handling the errors by asking "(2) *which of the solutions are identified by you as errors? Why? (3)What could be the sources of each error? (4) What could be the possible strategies to repair each error?*" The items 2(2), 2(3), and 2(4) with equal-group measurement division conducted at stage 2 are displayed in Figure 2, as an example.

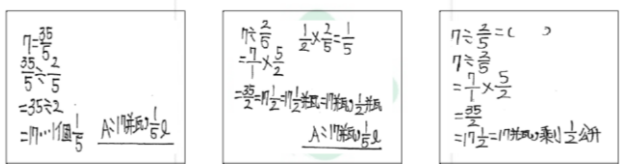
Data Analysis

A cross-cases analysis was used to analyze the four NTs' anticipating, identifying, and possible teaching strategies responding to students‘ errors. A content analysis was used to identify specific characteristics of the collected data. Coding the collected data into categories relevant to the research objectives is an essential procedure of a content analysis (Gall, Borg, Gall, 1996).

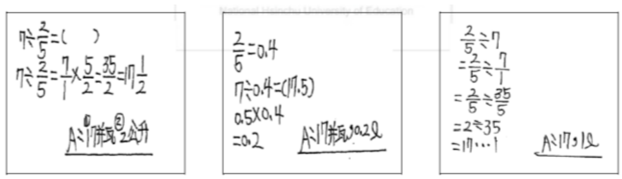
Item 2: Jing gave students to solve the problem:
A bucket of apple juice contains 7 liters. Every 2/5 liters is poured into a bottle. How many bottles of the juice are needed? What is the remainder of the apple juice?
9 solutions came up from students shown as follows.



1-(a) 1-(b) 1-(c)



1-(d) 1-(e) 1-(f)



1-(g) 1-(h) 1-(i)

(2) Which of the solutions you identified are errors? Why?
(3) What could be the sources of each error?
(4) What could be the possible strategies to respond to each error?

Figure 1. *The item 2(2), 2(3), and 2(4) conducted at the stage II in pre- and post-test.*

The data were analyzed by following three aspects of dividing fractions: (a) anticipating students’ possible errors; (b) detecting the causes of SE; and (c) suggesting pedagogical ways of repairing SE. Due to the limited pages, the analysis of anticipating SE and repairing SE are reported in the Result section.

Three categories consisted of 12 subcategories for anticipating students’ errors (AE) from the NT responses. Three categories included in conceptually based mistakes (AEA), two categories included in intuitively based mistakes (AEB), and four categories included in algorithmically based mistakes (AEC).

The subcategories of each category are described in Table 2.

Table 2
Coding Schema of Anticipating Students' Possible Errors

Novice teachers' anticipation of students' possible errors
AEA: Conceptually based mistakes
AEA1: error in deciding the dividend
AEA2: error in number sentence
AEA3: error in equivalent fraction
AEA4: error in deciding the remainder of division
AEA5: error in using dividing numerator and denominator strategy
AEA6: error in the remainder by using the "invert and multiply" method
AEB: Intuitively based mistakes
AEB1: divisor must be less than dividend
AEB2: quotient must be less than dividend
AEC: Algorithmically based mistakes
AEC1: no inversion of the divisor and multiply
AEC2: inverting the divisor and dividend
AEC3: inverting the dividend
AEC4: inverting the divisor and multiply but error in calculation

Eleven possible pedagogical strategies for repairing the errors (RE) were sorted into three categories, including (1) to help students to understand prerequisite knowledge of fraction division (REP), (2) to help students to understand the semantic structures of dividing fractions (RES), (3) to help students to understand the nature of divisions (REM). The subcategories of each category of repairing SE are depicted in Table 3.

Table 3
Coding Schema of Novice Teachers' Possible Pedagogical Strategies Repairing the Errors

Novice teachers' possible pedagogical strategies to repair the errors (RE)
REP: To revisit prior knowledge of fractions division
REP1: equivalent fractions
REP2: fraction as the result of division
REP3: fraction division with like denominators
REP4: ratio
RES: To understand the semantic structures of dividing fractions
RES1: equal-group partition division
RES2: equal-group measurement division
RES3: compare partition division
RES4: compare measurement division
REM: To understand the nature of divisions
REM1: the divisor can be larger than the dividend
REM2: the quotient can be larger than the dividend

REM3: “dividing numerator and denominator” does work

Results

Improvement of Anticipating Students' Errors

Three patterns were shown in Table 4, and Table 5 shows the comparison between pre- and post-tests on anticipating SE.

First, the NTs anticipated one possible error (18÷(4×4)=1.1) in pre-test increasing to 4 (63÷(4×4) ≒4) types of errors in post-test. Secondly, the errors they anticipated in the pre- or post-tests were mainly algorithmic mistakes rather than intuitive mistakes and conceptual misunderstanding. Moreover, the number of algorithmic errors they anticipated in the pre- and post-test were 9 and 31 respectively. Within the error in algorithm, the error in deciding the dividend (AEA1) is most common. For instance, they anticipated students might make the error $\frac{3}{4} \div \frac{8}{5}$ instead of $\frac{8}{5} \div \frac{3}{4}$ as the number sentence for the problem "*A bottle of water is $\frac{8}{5}$ liters; the water in a bottle is $\frac{3}{4}$ times as much as in a cup. How much water is in a cup?"*. It is noted that before cases discussion they anticipated students using $\frac{8}{5} \div \frac{15}{5} = \frac{8 \div 15}{5 \div 5} = \frac{8}{15}$. as an error type, the dividing numerator and denominator strategy (AEA5).

Second, Table 5 shows that the case discussion improved the NT's anticipation errors in intuition mistakes and in conceptually mistakes. The number of intuitive mistakes they anticipated in the pre- test was from 3 increased to 8 in the post-test. Likewise, the number of error in conceptual misunderstanding in the pre- test was from 6 increased to 24 in the post-test. Errors that might be related to deficient conceptual understanding included errors in writing the number sentence (AEA2), errors in equivalent fraction (AEA3), errors in the remainder of a division (AEA4), and errors in using the "invert and multiply" method to solve fraction division with a remainder (AEA6). For instance, errors in the remainder of a division (AEA4) is $\frac{1}{2}l$ instead of $\frac{1}{5}l$, since the solution "$7 \div \frac{2}{5} = 7 \times \frac{5}{2} = \frac{35}{2} = 17\frac{1}{2}$ for the problem: *A bucket of apple juice contains 7 liters. Every 2/5 of liters is poured into a bottle. How many bottles of the juice are need? What is the remainder of the apple juice?*".

The errors in algorithm anticipated by the NTs included three kinds of error types: errors in inverting the divisor first and then dividing (AEC2), errors in inverting the dividend (AEC3), and errors in inverting the divisor and multiply but the error in calculation (AEC4). Errors in inverting the dividend (AEC3) is like $\frac{8}{5} \div \frac{3}{4} = \frac{5}{8} \times \frac{3}{4}$. The "dividing numerators and denominators" strategy can only be applied when both the numerator and denominator of the dividend are divisible by those of the divisor (*e.g.* $\frac{36}{20} \div \frac{3}{4} = \frac{36 \div 3}{20 \div 4} = \frac{12}{5}$).

Third, the data in Table 5 shows the frequency of AEA5 is from 5 in the pre-test decreasing to 0 in the post-test. This indicates that the NTs did not agree the method "dividing numerators and denominators" in the pre-test. They eventually accepted the "dividing numerators and denominators" method as an algorithm of division of fractions. The case discussion contributes to this change. The casebook version was displayed in the casebook as follows.

Problem: A cup of milk with 8/5 liters is equally shared with 3 friends. How much of the milk will each friend get?

Shiang's solution: $\frac{8}{5} \div 3 = \frac{8}{5} \div \frac{15}{5} = \frac{8 \div 15}{5 \div 5} = \frac{8}{15}$, *since their denominators are the same, so that the answer is 8/15.*

Yuan disagrees with Shiang's method because the two denominators were divided by each other, the denominator became 1 instead of 15.

Do you agree with Shiang's or Yuan's method? If you were the case-teacher, then what would you do?

Table 4

Comparison of Nts' Anticipation of the Se in Pre- and Post-Test

	Conceptually based mistakes (AEA)						
	AEA1	AEA2	AEA3	AEA4	AEA5	AEA6	Total
Pre-test	1	0	0	0	5	0	6
Post-test	7	2	5	5	0	5	24
	Intuitively based mistakes (AEB)						
	AEB1	AEB2					Total
Pre-test	3	4					3
Post-test	0	4					8
	Algorithmically based mistakes (AEC)						
	AEC1	AEC2	AEC3	AEC4			Total
Pre-test	5	1	1	2			9
Post-test	11	7	7	6			31

The case discussion was excerpted as follows.

1. T0: What do you think of Shiang's solution? Do you think the method "dividing numerators and denominators" works?
2. T4: I do not accept this method. My students whom I taught when I was a substitute teacher used it for dividing fractions. I told them it is incorrect.
3. T3: So did I. The method $\frac{b}{a} \div \frac{d}{c} = \frac{b \div d}{a \div c}$ is similar to the multiplication of fractions $\frac{b}{a} x \frac{d}{c} = \frac{bxd}{axc}$. Students overgeneralized the algorithm of multiplication to the fraction division.
4. T0: There were several examples [$\frac{8}{10} \div \frac{2}{5}$; $\frac{8}{6} \div \frac{1}{2}$; $\frac{2}{4} \div \frac{1}{2}$]. Can you examine if each of them is correct?
5. All: Yes, but it only works on both the numerator and denominator of the dividend are divisible by those in the divisor [$\frac{b}{a} \div \frac{d}{c} = \frac{b \div d}{a \div c}$, $when\ d\ is\ a\ factor\ of\ ,\ c\ is\ a\ factor\ of\ a$.]
6. T0: Can we prove $\frac{b}{a} \div \frac{d}{c} = \frac{b \div d}{a \div c}$ by using algebraic expression?
7. T0: [wrote down the algebraic expressions on the blackboard]

$$\frac{b}{a} \div \frac{d}{c} = \frac{bc}{ac} \div \frac{ad}{ac} = \frac{bc}{ad}, \quad \frac{b \div d}{a \div c} = \frac{\frac{b}{d}}{\frac{a}{c}} = \frac{bc}{ad}, \text{ thus, } \frac{b}{a} \div \frac{d}{c} = \frac{b \div d}{a \div c}$$

8. So, it works for fraction division. (Case discussion meeting, 1005)

However, the NTs stated that they would not like to encourage their students to use the alternative method because it is too complex when both the numerator and denominator of the dividend are not divisible by those of the divisor.

Finally, the four teachers were not aware of the problem of fraction division with a remainder in the pre-test. Moreover, they were not aware of the error in calculating the remainder, through use of cases, five errors were anticipated by the NTs in the post-test. The result indicates that the NTs participating in the RBC workshops enhanced their knowledge of students' understanding, and then improved their anticipation of students' errors.

Improvement of Pedagogical Strategies of Repairing Students' Errors

After anticipating and detecting students' errors, the NTs were asked to figure out pedagogical strategies for repairing students' errors. The pedagogical strategies they suggested were sorted into three categories: (1) Revisiting prior knowledge of fraction division (REP), (2) enhancing the semantics structures of fractions (RES) for repairing the error caused from the understanding of the word problems, and (3) enhancing the nature of divisions for handling the errors caused from intuition (REM) (see Table 5).

Regarding repairing students errors in fraction division, NTs mentioned that they would like to revisit students' prior knowledge, including equivalent fractions (REP1), fraction as the result of division (REP2), fraction division with like denominators (REP3), and ratio (REP4). The semantic structures of fraction division word problems the NTs offered included equal-group partition division (RES1), equal-group measurement division (RES2), compare partition division (RES3), and compare measurement division (RES4) (see Table 1). The nature of fraction division included that the divisor can be larger than the dividend (REM1), the quotient can be larger than the dividend (REM2), and the method of "dividing numerator and denominator" can be true (REM3) (see Table 5).

Three major findings were shown in Table 6. First, the NTs figured out 28 possible pedagogical strategies for repairing SE in the pre- test. The number of strategies proposed by each NT is $28 \div (4\times 4) = 1.6$ in the pretest. After case discussion workshop, each NT was able to give more pedagogical strategies to handle the errors in the post-test. Each NT proposed $(68 + 16 + 24) \div (4\times 4) = 6.75$ pedagogical strategies of repairing the errors for each item. The data analysis shows that the workshop enhanced their knowledge of repairing students' errors.

Table 5

Comparison of Nts' Repairing Se in Pre- And Post-Test

	To review prior knowledge (REP)				
	REP1	REP2	REP3	REP4	Total
Pretest	0	0	3	25	28
Post-test	8	0	30	30	68
	To understand the semantic structures (RES)				
	RES1	RES2	RES3	RES4	Total
Pretest	0	0	0	0	0
Post-test	5	9	1	1	16
	To understand the nature of divisions in arithmetic operation (REM)				
	REM1	REM2	REM3		Total
Pretest	0	0	0		0
Post-test	2	6	16		24

Second, each NT not only understood the method of "dividing numerators and denominators", but also applied it to explain the remainder of fraction division. Third, in the post-test, when students encountered difficulties with solving the problem of fraction division, the NTs preferred to help students to revisit the prerequisite concepts of fraction division (RPE) (68 frequencies). In particular, revisiting the like denominators of fraction division. Helping students to review fraction division with like denominators (REP3) with 30 frequencies was the most frequent to be referred by the NTs' to repair students' errors. If students had difficulty with deciding the "remainder" of a fraction division, they would help students to revisit the knowledge of ratio (REP4, 30 frequencies). For instance, when the NTs were asked to repair the error "$7 \div \frac{2}{5} = \frac{7}{1} \times \frac{5}{2} = \frac{35}{2} = 17\frac{1}{2}$, the remainder is $\frac{1}{2}l$ "for the problem "*A bucket of apple juice contains 7 liters. Every 2/5 liters is poured into a bottle. How many bottles of the juice are need? What is the remainder of the apple juice?"*, they would use the ratio to help them to handle "the remainder $\frac{1}{2}l$". One of the NTs stated:

> I would like to use the strategy to help students to find out the remainder of a fraction division, since $7 \div \frac{2}{5} = \frac{35}{5} \div \frac{2}{5} = \frac{35}{5} \times \frac{5}{2} = 17\frac{1}{2}$, here $\frac{1}{2}$ means $\frac{1}{2}$ bottles rather than $\frac{1}{2}$ liter. Students need to be helped in $\frac{1}{2}$ bottle converting into "how many liters". The ratio needed to be used. The problem "1 bottle contains $\frac{2}{5}$ l, how many liters can be contained in 1/2 bottle "can be represented as 1: 2/5= 1/2 : ? The left is 2/5× 1/2 = 1/5 liter'' (Case discussion meeting, 1011223).

Finally, the result shows that in the pre-test, the NTs did not know how to deal with students' difficulty with deciding the dividend. In the post-test, they mentioned that they would like to help students to understand various situations: a divisor can be larger or smaller than a dividend expressed in a number sentence of fraction division.

The facilitator T0 asked the NTs why they would use the strategy of "division with like denominator" to deal with difficulties in deciding the "remainder" of division. The NTs were asked to explain the meaning of "inverting and multiplying". T3 explained that she would like to use the unitizing concept that "2/3 are made of two "1/3 "liters" to teach fraction division instead of "division as the inverse of multiplication". The discussion on "division with like denominator" strategy was excerpted as below. The scenario is about solving the problem: *A bucket of apple juice contains 7 liters. Every 2/5 liters is poured into a bottle. How many bottles of the juice are needed? What is the remainder of the apple juice?*

79. T0: Why would you like to use the strategy of "division with like denominator" to help students to find the remainder of a division?
80. T3: It is easier to explain the meaning of the remainder. For example, $7 \div \frac{2}{5} = \frac{35}{5} \div \frac{2}{5} = 35 \div 2 = 17\frac{1}{2}$, the remainder "$\frac{1}{2}$ bottle" students used was incorrect.
81. T0: Why the strategy can be used for explaining the meaning of the remainder?
82. T3: Here, a division with like denominator $\frac{35}{5} \div \frac{2}{5}$ can be explained by the unitizing concept. 2/3 liters are made of two "1/3" liters instead of by using the method of "division as the inverse of multiplication 〔$7 \div \frac{2}{5} = 7 \times \frac{5}{2}$〕" . $\frac{35}{5} \div \frac{2}{5}$ = 35 fifths divided by 2 fifths= (34 and 1) fifths divided by 2 fifths = (34 fifth divided by 2 fifths) and (1 fifth)= 17 bottles and 1/5liter , the answer is 17 bottle and $\frac{1}{5}$ *liter*".
83. T4: ……The problem "1 bottle contains $\frac{2}{5}l$, how many liters can be contained in 1/2 bottle" can be represented as 1: 2/5= 1/2 : ?. The remainder is 2/5× 1/2 = 1/5 liter.
84 T0: Which of the strategies do you prefer to use?
85 T2: It depends on the type of word problems. If the word problem is fraction division with a remainder, then I would use the "unitizing" concept. Otherwise, I would prefer to use the "division as the inverse of multiplication" strategy (Case discussion meeting, 1011223).

Discussions and Conclusions

This study shows that the NTs participating in the study enhanced their knowledge of students' learning of fraction division through the use of case discussion. The knowledge of students' learning included three aspects, anticipating students' errors, detecting the causes of errors, and repairing the errors. Before participating in the case discussion, the NTs did not accept the "dividing numerators and denominators" method. Through the use of case discussion, they changes their knowledge. However, they would not recommend this method to their students, since it is hard for the elementary students to handle when both the numerator and denominator of the dividend in fraction division are not divisible by those of the divisor.

It is obvious that the case discussion helped the novice teachers to realize that the word problem of fraction divisions with a remainder was an appropriate type of problems to check if students understood the meaning of the "inverting and multiplying" algorithm. Before case discussion, the NTs were not aware of a remainder of fraction division is one type of word problem of fraction division. They also had difficulty with repairing students' error in deciding how to deal with "the remainder" of fraction division. After case discussion, they learned "unitizing" concept as a pedagogical strategy for repairing this difficulty or error. They agreed that unitizing is an appropriated method to decide the value of "a remainder" of fraction division with like denominators, as a prerequisite knowledge of the fraction division. While facing students' various errors of fraction division, the NTs preferred to help students to revisit the prerequisite knowledge of fraction division. The other is that the NTs would help students to understand the meaning of the dividend and the divisor from various semantic structures.

Two possible factors of improving NTs' knowledge of students' errors were discussed here: (1) The students' errors in the written cases were authentic rather than imagined. Through the use of cases, the NTs had the opportunity to foresee the errors patterns. The finding of the study supports Fennema et al. (1996) who claimed that improving knowledge of students' conceptions and misconceptions is not static and can probably only be acquired in the context of teaching; (2) The facilitator plays a role of asking critical questions to make the NTs' reflect to their own experience of learning or teaching mathematics. In the case discussion, the facilitator provided more opportunities to invite more effective pedagogical strategies from in-service experienced teacher for dealing with students' errors in the topic of fraction division. Therefore, the case discussion initiating social interaction between experienced teachers and novice teachers was a possible factor in improving the NTs' knowledge of students' errors of dividing fractions.

By exploring what novice teachers knew about students' errors on dividing fractions, this study suggests that the teacher educators should provide the NTs or pre-service teachers' with students' authentic errors or difficulties on any mathematics contents in teacher preparation program. In addition, this study also suggests to improve novice teachers' knowledge of students learning starting from anticipating students' possible errors, and proposing possible pedagogical repairing errors, and further to examine how the suggested pedagogical strategies work in the real classrooms.

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