Journal of Mathematics Education © *Education for All*
July 2017, Vol. 10, No. 1, pp. 76-91 *DOI: https://doi.org/10.26711/007577152790006*

Fostering Novice Teachers' Knowledge of Students' Errors on Fraction Division by Using Researched-Based Cases

Pi-Jen Lin
National Tsing Hua University (Nanda Campus), Taiwan, R. O. C.

There are increasing studies on knowledge of students' learning, but limited study focuses on detecting students' errors gathered in authentic classrooms and repairing them. Novice primary teachers, as well as students, have difficulties with dividing fractions and have weak knowledge of students' understanding and misunderstanding. Thus, this study was designed to foster novice primary teachers' knowledge of anticipating, identifying, and correcting students' errors on dividing fractions through using research-based cases. The data were collected from a case-based discussion group including four novice teachers and a facilitator. Five case-based discussions were tape- and video-recorded. A pre- and post-test consisting of four items with 16 sub-items were carried out. The result shows that the novice teachers became more versatile in anticipating students' errors and shifted from algorithms based errors toward conceptually based errors. The pedagogical strategies of repairing students' errors to be used included to help students revisit their prior knowledge, understand semantics structure of fraction division word problems, and realize the nature of fraction division. They were able to offer various pedagogical strategies corresponding to specific errors to repair students' errors.

***Keywords*:** Research-based, novice teachers, teacher's knowledge of students' errors, fraction division.

Fraction is considered as a complex mathematical content in elementary school mathematics (Mack, 1990). The complexity results into more difficulty on fractions than other topics for students and teachers (Rizvi & Lawson, 2007; Tirosh, 2000). Particularly, pre-service and novice primary teachers have difficulties with dividing fractions (Tirosh, 2000). Research suggests that novice teachers lack of deep understanding of fraction division and have weak knowledge of students' misunderstanding (Young & Zientek, 2011)

Traditionally, teachers hide and ignore students' mathematical mistakes and make students afraid to make mistakes (Durkin & Rittle-Johnson, 2015). If a teacher ignores students' errors (SE) or does not correct student's error, the error may be repeated or accumulated or cause other related errors. Nevertheless, the detection of errors is a pre-condition for revising false knowledge, and therefore preventing the errors from reoccurring. Research shows that some of in-service teachers are short of knowledge to identify SE and the causes of the errors of dividing fractions (Tirosh, 2000). Novice teachers with little teaching experience have weaker knowledge of students' errors and pedagogically repair the errors of fractions (e.g., Lo & Luo, 2012). Thus, novice teachers' (NTs) about students' learning fraction division should be enhanced.

Fennema, Carpenter, Franke, Levi, Jacobs, and Empson (1996) suggest that improving knowledge of students' misconceptions can probably only be acquired in the context of teaching. This implies that teachers should learn about students' difficulties via classroom teaching. However, it is difficult for NTs to use SE as a teaching tool without similar learning experience in teacher preparation. The use of cases appears to be a way of fostering NTs to develop this knowledge, since the cases include various errors of a specific content collected from authentic errors.

This study focuses on teachers' identification of students' errors gathered in realistic classrooms. It aims to examine the effect of the research-based cases on helping novice primary teachers to identify the students' errors, their causes, and figure out possible strategies to repair the errors. Two research questions consisted of: (1) What are the students' errors of fraction division the novice teachers identified before and after the case discussion workshops? (2) What are the pedagogical repairs of students' errors of fraction division the novice teachers suggested before and after the case discussion workshops?

Theoretical Background

The Importance of Knowledge of Students' Errors

There are several aspects of knowledge of students' errors or misconceptions, such as anticipating and diagnosing students' errors. Diagnosing errors is important for understanding students' changing knowledge (Durkin & Rittle-Johnson, 2015). The importance of knowledge of students' errors (KSE) can be illustrated by Minsky's (1994) theory of two complementary types of knowledge: negative and positive knowledge. Negative knowledge means knowing about incorrect concepts and procedures. We are not usually taught incorrect concepts or procedures but we sometimes have errors or misconceptions. The Chinese slogan "failure is a mother of success" means the use of errors can be achieved into success, since errors can be used as a resource of detecting the causes of error or as resources for promoting learning (Borasi, 1994; Heinze & Reiss, 2007). Misconceptions or errors can persist over a long time and must be overcome (Eryilmaz, 2002). Hence, one can learn from handling errors. Teaching to hide misconceptions could result in misconceptions, which might be hidden from the teacher and from the children themselves. In contrary, if there is clear feedback or dealing with errors during the learning process, it seems to be more effective than hiding it (Keith & Frese, 2008). If a teacher is familiar with various causes of errors in children's responses, she/he would be likely to select an appropriate

The Research-Based Cases Workshops

Six 3-hours RBC workshops related to fraction division were held within a semester. Five written cases from a casebook were discussed in the six RBC workshops. Each written case consists of seven sections: Introduction of the case, Grade level, Prior knowledge, Objectives of the case, Context of the case including problems that were posed, Dialogues between teacher and students, Students' authentic correct or incorrect solutions, Discussion questions, and User's guides.

Case 1 demonstrates students' strategy "dividing the numerators and denominators" for solving the type of $\frac{b}{a} \div c$ in the context of partition division. For example, *A cup of Coke with 6/5 liters is equally shared with 4 friends. How much of the Coke will each friend get?* Case 2 describes students' strategies for solving the type of ($\frac{b}{a} \div \frac{d}{c}$, c a factor of a) in the context of comparison measurement division (Lo & Luo, 2012). Case 3 displays students' nine solutions of solving the problem $\left(a \div \frac{c}{b} \right)$ without a remainder in the context of measurement division. Case 4 displays students' nine solutions of solving the problem $\left(a \div \frac{c}{b} \right)$ with a remainder under the context of measurement division. For example, *A bucket of paint contains 5 liters. Every 2/3 of liters is poured into a bottle. How many bottles of the paint are needed? What is the remainder of the paint?* Case 5 describes students' strategies for solving the type of ($\frac{b}{a} \div \frac{d}{c}$) in the context of comparison of partition division and the algorithm of inverting and multiplying for the fraction division. For example, *a bag of rice weighs 12/16kg. A bag of beans weighs 1/4kg. How much does the bag of rice weigh as compared to the bag of beans?*

Data Collection

Prior to each workshop, all participants were required to read an assigned written case in advance. Each RBC workshop started with one-hour small group discussion. During discussion, the participants were also commonly asked to answer the questions: (1) What is the main mathematical idea in the case? (2) Which of the solutions are incorrect? (3) What might be the causes of the errors listed in the case? (4) What pedagogical strategies you would like to respond to each error? The workshops were video recorded.

The participants were administered pre-test and post-test on SE in fraction division. Two dimensions were involved in the framework of developing the items (see Table 1). One dimension was knowledge of SE of dividing fractions, consisting of anticipating students' possible errors (AE), detecting the errors given in solutions (DE), and handling the errors (RE). The other dimension was the semantic structure of fraction division, consisting of equal-group partition division (EPd), equal-group measurement division (EMd), comparison of measurement division (CMd), and comparison of partition division (CPd). We also consider the remainder is nonzero.

Table 1
Framework of Pre- and Post-Test

Knowledge of students' errors / Semantic structures	AE	DE	CE	RE
Equal-group partition division(EPd) (*A cup of milk with 8/5 liters is equally shared with 3 friends. How much of the milk will each friend get?*)	Item 1(1)	Item 1(2)	Item 1(3)	Item 1(4)
Equal-group measurement division (EMd) (*A bucket of apple juice contains 7 liters. Every 2/5 liters is poured into a bottle. How many bottles of the juice are needed? What is the remainder of the apple juice?*)	Item 2(1)	Item 2(2)	Item 2(3)	Item 2(4)
Comparison partition division (CPd) (*A bag of green beans weighs 16/9kg. A bag of red beans weighs 1/3kg. How much does the bag of green beans weigh as compared to the bag of red beans?*)	Item 3(1)	Item 3(2)	Item 3(3)	Item 3(4)
Comparison measurement division (CMd) (*A bottle of water is 8/5 liters; the water in a bottle is 3/4 times more than in a bucket. How much water is in a bucket?*)	Item 4(1)	Item 4(2)	Item 4(3)	Item 4(4)

AE: Anticipating possible errors DE: Detecting errors CE: Interpreting causes RE: Repairing errors

The pre-test and post-test were administrated in two stages. Post-test apart five months from pre-test with the same items were conducted.

Stage I is to anticipate students' possible errors by asking "*what are possible errors or mistakes students may make when solving the problem?*" An item 2(1) with the context of EMd is shown as follows.

> "*A bucket of apple juice contains 7 liters. Every 2/5 of liters is poured into a bottle. How many bottles of the juice are needed? What is the remainder of the apple juice?*
> *(1) what are possible errors or mistakes students may make when solving the problem?*"

Stage II is to explore teachers' knowledge of detecting students' errors and handling the errors by asking "(2) *which of the solutions are identified by you as errors? Why? (3)What could be the sources of each error? (4) What could be the possible strategies to repair each error?*" The items 2(2), 2(3), and 2(4) with equal-group measurement division conducted at stage 2 are displayed in Figure 2, as an example.

Data Analysis

A cross-cases analysis was used to analyze the four NTs' anticipating, identifying, and possible teaching strategies responding to students' errors. A content analysis was used to identify specific characteristics of the collected

Table 2

Coding Schema of Anticipating Students' Possible Errors

Novice teachers' anticipation of students' possible errors
AEA: Conceptually based mistakes
AEA1: error in deciding the dividend
AEA2: error in number sentence
AEA3: error in equivalent fraction
AEA4: error in deciding the remainder of division
AEA5: error in using dividing numerator and denominator strategy
AEA6: error in the remainder by using the "invert and multiply" method
AEB: Intuitively based mistakes
AEB1: divisor must be less than dividend
AEB2: quotient must be less than dividend
AEC: Algorithmically based mistakes
AEC1: no inversion of the divisor and multiply
AEC2: inverting the divisor and dividend
AEC3: inverting the dividend
AEC4: inverting the divisor and multiply but error in calculation

Eleven possible pedagogical strategies for repairing the errors (RE) were sorted into three categories, including (1) to help students to understand prerequisite knowledge of fraction division (REP), (2) to help students to understand the semantic structures of dividing fractions (RES), (3) to help students to understand the nature of divisions (REM). The subcategories of each category of repairing SE are depicted in Table 3.

Table 3

Coding Schema of Novice Teachers' Possible Pedagogical Strategies Repairing the Errors

Novice teachers' possible pedagogical strategies to repair the errors (RE)
REP: To revisit prior knowledge of fractions division
REP1: equivalent fractions
REP2: fraction as the result of division
REP3: fraction division with like denominators
REP4: ratio
RES: To understand the semantic structures of dividing fractions
RES1: equal-group partition division
RES2: equal-group measurement division
RES3: compare partition division
RES4: compare measurement division
REM: To understand the nature of divisions
REM1: the divisor can be larger than the dividend
REM2: the quotient can be larger than the dividend
REM3: "dividing numerator and denominator" does work

eventually accepted the "dividing numerators and denominators" method as an algorithm of division of fractions. The case discussion contributes to this change. The casebook version was displayed in the casebook as follows.

Problem: A cup of milk with 8/5 liters is equally shared with 3 friends. How much of the milk will each friend get?

Shiang's solution: $\frac{8}{5} \div 3 = \frac{8}{5} \div \frac{15}{5} = \frac{8 \div 15}{5 \div 5} = \frac{8}{15}$, *since their denominators are the same, so that the answer is 8/15.*

Yuan disagrees with Shiang's method because the two denominators were divided by each other, the denominator became 1 instead of 15.

Do you agree with Shiang's or Yuan's method? If you were the case-teacher, then what would you do?

Table 4

Comparison of Nts' Anticipation of the Se in Pre- and Post-Test

	Conceptually based mistakes (AEA)						
	AEA1	AEA2	AEA3	AEA4	AEA5	AEA6	Total
Pre-test	1	0	0	0	5	0	6
Post-test	7	2	5	5	0	5	24
	Intuitively based mistakes (AEB)						
	AEB1	AEB2					Total
Pre-test	3	4					3
Post-test	0	4					8
	Algorithmically based mistakes (AEC)						
	AEC1	AEC2	AEC3	AEC4			Total
Pre-test	5	1	1	2			9
Post-test	11	7	7	6			31

The case discussion was excerpted as follows.

1. T0: What do you think of Shiang's solution? Do you think the method "dividing numerators and denominators" works?
2. T4: I do not accept this method. My students whom I taught when I was a substitute teacher used it for dividing fractions. I told them it is incorrect.
3. T3: So did I. The method $\frac{b}{a} \div \frac{d}{c} = \frac{b \div d}{a \div c}$ is similar to the multiplication of fractions $\frac{b}{a} x \frac{d}{c} = \frac{bxd}{axc}$. Students overgeneralized the algorithm of multiplication to the fraction division.
4. T0: There were several examples [$\frac{8}{10} \div \frac{2}{5}$; $\frac{8}{6} \div \frac{1}{2}$; $\frac{2}{4} \div \frac{1}{2}$]. Can you examine if each of them is correct?
5. All: Yes, but it only works on both the numerator and denominator of the dividend are divisible by those in the divisor [$\frac{b}{a} \div \frac{d}{c} = \frac{b \div d}{a \div c}$, $when\ d\ is\ a\ factor\ of\ ,\ c\ is\ a\ factor\ of\ a$.]
6. T0: Can we prove $\frac{b}{a} \div \frac{d}{c} = \frac{b \div d}{a \div c}$ by using algebraic expression?
7. T0: [wrote down the algebraic expressions on the blackboard]

Table 5
Comparison of Nts' Repairing Se in Pre- And Post-Test

	To review prior knowledge (REP)				
	REP1	REP2	REP3	REP4	Total
Pretest	0	0	3	25	28
Post-test	8	0	30	30	68
	To understand the semantic structures (RES)				
	RES1	RES2	RES3	RES4	Total
Pretest	0	0	0	0	0
Post-test	5	9	1	1	16
	To understand the nature of divisions in arithmetic operation (REM)				
	REM1	REM2	REM3		Total
Pretest	0	0	0		0
Post-test	2	6	16		24

Second, each NT not only understood the method of "dividing numerators and denominators", but also applied it to explain the remainder of fraction division. Third, in the post-test, when students encountered difficulties with solving the problem of fraction division, the NTs preferred to help students to revisit the prerequisite concepts of fraction division (RPE) (68 frequencies). In particular, revisiting the like denominators of fraction division. Helping students to review fraction division with like denominators (REP3) with 30 frequencies was the most frequent to be referred by the NTs' to repair students' errors. If students had difficulty with deciding the "remainder" of a fraction division, they would help students to revisit the knowledge of ratio (REP4, 30 frequencies). For instance, when the NTs were asked to repair the error "$7 \div \frac{2}{5} = \frac{7}{1} \times \frac{5}{2} = \frac{35}{2} = 17\frac{1}{2}$, the remainder is $\frac{1}{2}l$ "for the problem "*A bucket of apple juice contains 7 liters. Every 2/5 liters is poured into a bottle. How many bottles of the juice are need? What is the remainder of the apple juice?"*, they would use the ratio to help them to handle "the remainder $\frac{1}{2}l$". One of the NTs stated:

> I would like to use the strategy to help students to find out the remainder of a fraction division, since $7 \div \frac{2}{5} = \frac{35}{5} \div \frac{2}{5} = \frac{35}{5} \times \frac{5}{2} = 17\frac{1}{2}$, here $\frac{1}{2}$ means $\frac{1}{2}$ bottles rather than $\frac{1}{2}$ liter. Students need to be helped in $\frac{1}{2}$ bottle converting into "how many liters". The ratio needed to be used. The problem "1 bottle contains $\frac{2}{5}$ l, how many liters can be contained in 1/2 bottle "can be represented as 1: 2/5= 1/2 : ? The left is 2/5× 1/2＝1/5 liter'' (Case discussion meeting, 1011223).

Finally, the result shows that in the pre-test, the NTs did not know how to deal with students' difficulty with deciding the dividend. In the post-test, they mentioned that they would like to help students to understand various situations: a divisor can be larger or smaller than a dividend expressed in a number sentence of fraction division.

The facilitator T0 asked the NTs why they would use the strategy of "division with like denominator" to deal with difficulties in deciding the "remainder" of division. The NTs were asked to explain the meaning of "inverting and multiplying". T3 explained that she would like to use the unitizing concept that "2/3 are made of two "1/3 "liters" to teach fraction division instead of "division as the inverse of multiplication". The discussion on "division with like denominator" strategy was excerpted as below. The scenario is about solving the problem: *A bucket of apple juice contains 7 liters. Every 2/5 liters is poured into a bottle. How many bottles of the juice are needed? What is the remainder of the apple juice?*

79. T0: Why would you like to use the strategy of "division with like denominator" to help students to find the remainder of a division?
80. T3: It is easier to explain the meaning of the remainder. For example, $7 \div \frac{2}{5} = \frac{35}{5} \div \frac{2}{5} = 35 \div 2 = 17\frac{1}{2}$, the remainder "$\frac{1}{2}$ bottle" students used was incorrect.
81. T0: Why the strategy can be used for explaining the meaning of the remainder?
82. T3: Here, a division with like denominator $\frac{35}{5} \div \frac{2}{5}$ can be explained by the unitizing concept. 2/3 liters are made of two "1/3" liters instead of by using the method of "division as the inverse of multiplication 〔$7 \div \frac{2}{5} = 7 \times \frac{5}{2}$〕" . $\frac{35}{5} \div \frac{2}{5}$ = 35 fifths divided by 2 fifths= (34 and 1) fifths divided by 2 fifths = (34 fifth divided by 2 fifths) and (1 fifth)= 17 bottles and 1/5liter , the answer is 17 bottle and $\frac{1}{5}$ $liter$".
83. T4: ……The problem "1 bottle contains $\frac{2}{5}l$, how many liters can be contained in 1/2 bottle" can be represented as 1: 2/5= 1/2 : ?. The remainder is 2/5× 1/2＝1/5 liter.
84 T0: Which of the strategies do you prefer to use?
85 T2: It depends on the type of word problems. If the word problem is fraction division with a remainder, then I would use the "unitizing" concept. Otherwise, I would prefer to use the "division as the inverse of multiplication" strategy (Case discussion meeting, 1011223).

Discussions and Conclusions

This study shows that the NTs participating in the study enhanced their knowledge of students' learning of fraction division through the use of case discussion. The knowledge of students' learning included three aspects, anticipating students' errors, detecting the causes of errors, and repairing the errors. Before participating in the case discussion, the NTs did not accept the "dividing numerators and denominators" method. Through the use of case discussion, they changes their knowledge. However, they would not recommend this method to their students, since it is hard for the elementary students to handle when both the numerator and denominator of the dividend in fraction division are not divisible by those of the divisor.

Borasi, R. (1994). Capitalizing on errors as "springboards for inquiry": A teaching experiment. *Journal for Research in Mathematics Education, 25(2),* 166-208.

Borko, H., Eisenhar, M., Brown, C., Underhill, R., Jones, D., & Agard, P. (1992). Learning to teach hard mathematics: Do NTs and their instructors give up too easily? *Journal for Research in Mathematics Education, 23,* 194–222. doi:10.2307/749118.

Durkin, K., & Rittle-Johnson, B. (2015). Diagnosing misconceptions: Revealing changing decimal fraction knowledge. *Learning and Instruction, 37,* 21–29.

Eryilmaz, A. (2002). Effects of conceptual assignments and conceptual change discussions on students' misconceptions and achievement regarding force and motion. *Journal of Research in Science Teaching, 39,* 1001-1015.

Fennema, E. Carpenter, T. P., Franke, K. L., Levi, L., Jacobs, V. R. & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal of Research in Mathematics Education, 27*(4), 403-434.

Gall, M. D., Borg, W. R., & Gall, J. P. (1996). *Educational research: An introduction.* New York, NY: Longman.

Heinze, A.. & Reiss, K. (2007). Mistake-handling activities in the mathematics classroom: Effects of an inservice teachers training on students' performance in geometry. *Proceedings of the 31th Conference of the International Group for the Psychology of Mathematics Education* (Vol.3, pp.9-16). Korea, Soul University.

Isik, C., & Kar, T. (2012). An error analysis in division problems in fractions posed by pre-service elementary mathematics teachers. *Educational Sciences: Theory and Practice, 12*(3), 2303-2309.

Isiksal, M., & Cakiroglu, E. (2008). Preservice teachers' knowledge of students' cognitive processes about the division of fractions. *H. U. Journal of Education, 35,* 175-185.

Keith, N., & Frese, M. (2008). Effectiveness of error management training: A meta-analysis. *Journal of Applied Psychology, 93*(1), 59-69.

Li, Y., & Kulm, G. (2008). Knowledge and confidence of prospective mathematics teachers: The case of fraction division. *ZDM–The International Journal on Mathematics Education, 40,* 833–843.

Lin, P. J. (2010). Research-based case discussion supporting teachers on the use of curriculum materials. In M. M. F. Pinto & T. F. Kawaski (Eds.), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education* (Vol.4. 372). Belo Horizonte, Brazil: Federal University of Minas Gerais.

Lin, P.J. (2005). Using research-based video-cases to help pre-service primary teachers conceptualize a contemporary view of mathematics teaching. *International Journal of Mathematics and Science Education. 3*(3), 351-377.