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Teaching Arithmetic of Fractions Using Geometry

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Whole numbers are rather easy to work with. Most children seem to have fun counting toys, tools and treats and playing with whole numbers. But this is not true in general, for fractions. The question is: Why? Most arithmetic operations with whole numbers may be clearly explained through visual displays. When children see toys, often they also see the numbers. When they add, subtract, multiply and divide their toys, they truly visualize these operations and that makes arithmetic operations with whole numbers understandable and often enjoyable. For many children this pleasant environment of learning ceases to exist when they learn arithmetic operations with fractions. While some very elementary concepts of fractions could be easily explained by simple geometry, most concepts of arithmetic operations with fractions are often clouded with complications in the eyes of children. In this article, we have attempted to clear some these clouds by introducing visual geometrical understanding of such operations. Several illustrations have been explained.

Key words: whole number, teaching and learning,

Introduction

Arithmetic is the gate of the universe of mathematics. All children must master all the principles of arithmetic very thoroughly. This is the fundamental topic that all will need to use all of life. Learning arithmetic begins with counting natural numbers, understanding the use of zero from where everything got started and doing various calculations with them. Most children seem to enjoy all these. Often they overcome challenges imposed by word problems. Their thoughts are spontaneous and their logic is fun.

Concepts of arithmetic of fractions often ruin their mirth and interest regarding learning mathematics. They start getting disinterested in a topic that once they liked very much. And we strongly believe that this is not the fault of the children. Books and teachers must share some blame. As teachers of mathematics we wondered what do we do wrong that turns off a machinery of spontaneous interest of children toward mathematics. What do we do wrong for which many children consider mathematics as a bug bear. Working closely with children privately, we observed their psychology towards learning fractions is inspiring, whereas psychology towards memorizing all the rules and regulations regarding arithmetic operations with fractions is depressing. The primary reason for that is that envisioning how all the rules could be translated into pictures of understanding are not properly understood by them when they read the math book or take lessons on fractions in the school.

We cannot fully blame the teachers either. Most of them are totally aware of the difficulties that the children encounter while learning fractions. Often they use and test different techniques to teach fractions. But it becomes increasingly difficult for the children to conceptualize representations of the same fractions in a number of different forms of fractions. We worked with children on a one-to-one basis. Most of them could comprehend why $\frac{1}{2}$ is the same as $\frac{4}{8}$ or $\frac{18}{36}$. But why $\frac{3}{5}$ is the same as $\frac{6}{10}$, they failed to visualize, because then cannot quickly see these fractions in terms of some concrete objects like pieces of pizzas or cakes, etc. They try to memorize that $\frac{6}{10}=\frac{3\times 2}{5\times 2}=\frac{3}{5}$. Starting from this point, they feel that the entire lesson on fractions deal with a set of memorization of certain rules without which they will not be able to compute the correct answers. This is dreadful. This process of memorization without comprehension destroys their enthusiasm to appreciate mathematics. Starting from the lessons on fractions, many students develop their apathy toward mathematics. This dreadful trend must be stopped.

We fully agree with the author of “Why are fractions so difficult to learn for kids?” in the website, where Hunting (1999) stated “If children simply try to memorize these [rules of fractions] without knowing where they came from, they will probably seem like a jungle of seemingly meaningless rules. By meaningless I mean that the rule does not seem to connect with anything about the operation--it is just like a play where in each case you multiply or divide or add or do various things with the numerators and denominators and that then should give you the answer.” The author put emphasis on visualization of fractions. However, the elementary examples given in the text as seen on the website, do not address the issues on how to visualize fractional arithmetic rules geometrically.

In the San Antonio Home Education section of the website, Mack, 1998), developed by ERIC Digest, we read the article "Teaching Fractions: New Methods, New Resources" where works of several authors (e.g. Meagher, 2002) were analyzed. We have tried all these and found that children have often experienced considerable difficulties to master these concepts. However, we believe that if they could spend enough time working on fractions with these schemes, they might be successful.

Our approach is different from all of these. We have not only defined fractions geometrically, but all the various rules of arithmetic of fractions applying one uniform rule of geometrical representation. Our students did not have to memorize any special rule, because one uniform concept of geometry made that rule easily understandable. For instance, while adding or subtracting fractions, the concept of having a common denominator will spontaneously be clarified in their minds, just be observing the geometry applied to represent such operations. They need no software or any special toy to learn these operations. However, teachers with innovative ideas may easily develop their own software to display our method of arithmetic operations with fractions geometrically. That will certainly make their teaching more effective especially in this era where children seem to be more attracted by technology. Charlie Dey has started developing such a software where geometrical concepts of fractional arithmetic, as presented here could be easily visible in colors. The software is user-friendly and could be easily mastered by children.

We prefer to begin with some simplistic concepts of geometry of fractions and then extend these to arithmetic operations. We have made every attempt to write this article in such a way so that any teacher of mathematics starting from the kindergarten level could easily understand all that we have attempted to explain. Let us begin with some simple concepts on fractions. In all the examples of fractions common factors were not crossed out. This is done intentionally so that when these examples will be presented to the students they will use their imagination and follow the logic. We also wanted to be as informal as possible, representing our ideas.

Fractions are results from applications of division. Laman (2005) stated, "No one knows better than teachers who have had experience teaching fractions that current instruction is not serving many students. However, in addition to having a need to change, there must be a viable direction for change." She further stated, "Fraction, ratio and other multiplicative ideas are psychologically and mathematically complex… ”.

One primary reason is: This happens because of the absence of proper visual tools and techniques to understand arithmetic operations of fractions. Therefore it is very essential that some visual techniques should be developed in order to teach arithmetic effectively.

As we have mentioned before, arithmetic operation of division created fractions. Thus any visual technique to explain the concept of fraction must deal with division. For ages, such attempts have been launched. Cakes, cookies, pizzas were used as visual tools.

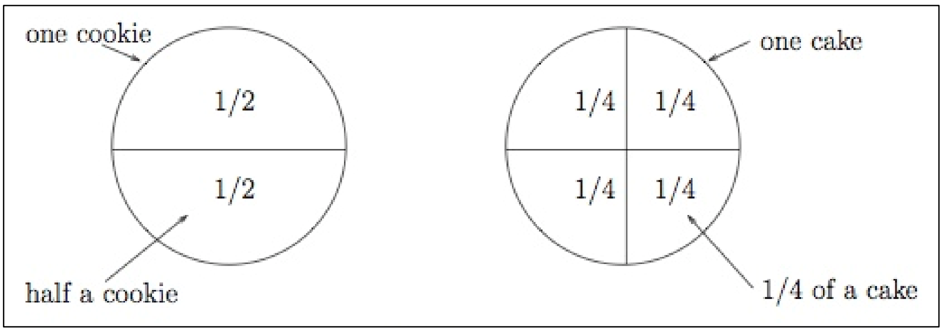

Figure 1. **Fraction 1.**

Children understand what is $\frac{1}{2}$ what is $\frac{1}{4}$, that two $\frac{1}{4}$'s make one $\frac{1}{2}$etc. by slicing a round cake. There are many such concepts which could be made very clear by slicing cakes or breaking cookies. Unfortunately, these techniques did not go any farther than making some simple solutions of some simple concepts regarding fractions.

Here we have attempted to explain geometry of the arithmetic of fractions. It is evident from the figures that two slices of $\frac{1}{4}$ this make $\frac{1}{2}$ and three such slices make $\frac{3}{4}$th of the cake and four slices make the full cake which is just one cake. That means $2\times\frac{1}{4}=\frac{1}{2}$ and $3\times\frac{1}{4}=\frac{3}{4}$ and $4\times\frac{1}{4}=1$. The concepts of $\frac{5}{4}$ or $\frac{9}{4}$ may simply be explained as: $\frac{5}{4}$=5 slices of $\frac{1}{4}^{th}$ =4 slices of $\frac{1}{4}$+one slice of $\frac{1}{4}$= 1 cake + $\frac{1}{4}^{th}$ of the cake=$1\frac{1}{4}$. And $\frac{9}{4}$=9 slices of $\frac{1}{4}^{th}$=4 slices of $\frac{1}{4}^{th}$+4 slices of $\frac{1}{4}$+one slice of $\frac{1}{4}$=2 cakes+$\frac{1}{4}^{th}$ of a cake = $2\frac{1}{4}$.

These examples are well-known. Seeing them, children understand some of the fundamentals of fractions. But these examples are not strong enough to elucidate all the rules of the arithmetic of fractions.

Visual Understanding of Fractional Arithmetic

Instead of drawing circles, let us draw rectangles to understand some of the concepts discussed in the previous section. After having clear concepts of fractions through rectangles, we will discuss how all the rules of arithmetic could be made easy to understand.

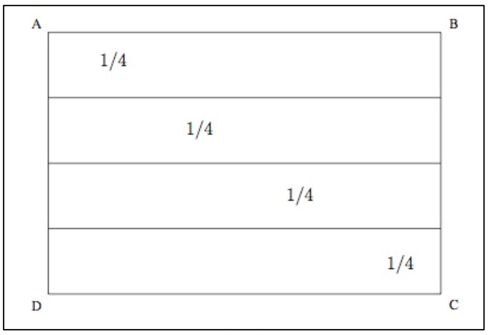

Figure 2. **Fraction 2.**

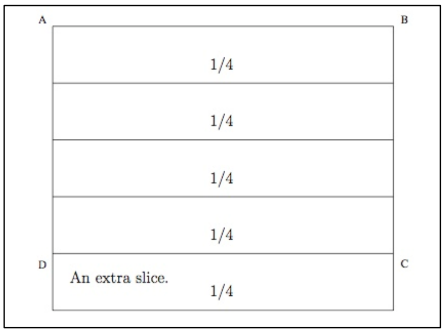

Figure 3. **Fraction 3.**

Let us think of a rectangular pizza *ABCD* sliced horizontally into 4 equal parts. Four kids can share the pizza equally each having $\frac{1}{4}$h of the pizza. If another kid drops in and wants an equal share, another $\frac{1}{4}$th slice must be ordered. Now there are 5 slices each equals 1/4th pizza. Clearly $\frac{5}{4} = 1 + \frac{1}{4} = 1\frac{1}{4}$(Figure 3).

Similarly 8 pieces of $\frac{1}{4}$th pizza = 2 pizzas and $\frac{9}{4}$=9 pieces of $\frac{1}{4}$th $=2\frac{1}{4}$, etc.. We will now extend these concepts to study inequalities, addition, subtraction, multiplication and division of fractions.

Inequality

Let us compare which of the inequalities $\frac{2}{3}, \frac{4}{5}$ is larger?

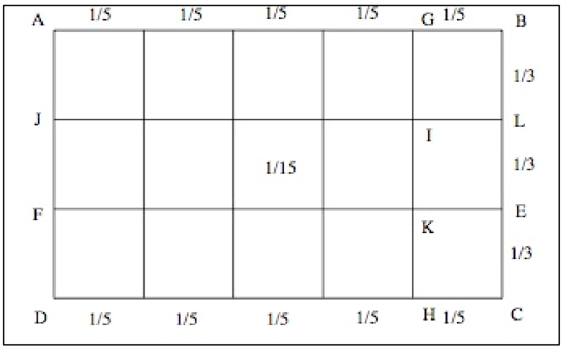

Figure 4. **Fraction 4.**

We have sliced a rectangle, $ABCD$, first into three equal parts horizontally so that each slice is $\frac{1}{3}$, and then vertically into 5 equal parts so that each vertical slice is $\frac{1}{5}$. There are 15 small rectangles in all. Thus our one big rectangle is equal to 15 small rectangles, each of which is equal to $\frac{1}{15}$.

Taking first two horizontal slices we get $\frac{2}{3}$(two $\frac{1}{3}$rd slices) of the rectangle $ABCD = ABEF$ consisting of 10 small rectangles. Considering first four vertical slices we get $\frac{4}{5}$ (four $\frac{1}{5}$th slices) of $ABCD = AGHD$ consisting of 12 small rectangles.

Thus, $\frac{4}{5}$ has more rectangles that $\frac{2}{3}$. Therefore, $\frac{4}{5} > \frac{2}{3}$. Notice that if we cross multiply we get $4 \times 3 > 2 \times 5$ giving $12 > 10$, giving, # of rectangles in $AGHD >$ # of rectangles in $ABEF$ which is exactly what we did while counting the small rectangles. This principle may be generalized. If $a \cdot d > b \cdot c$ then, $\frac{a}{b} > \frac{c}{d}$. At the present time we consider *a, b, c, d* to be positive integers.

Addition and Subtraction

Let us find what is $\frac{2}{3}+\frac{4}{5}$. From the Fig. 4, $\frac{2}{3}+\frac{4}{5}=\frac{2}{3}$rd of $ABCD$+$\frac{4}{5}$ th of $ABCD$. That means $\frac{2}{3}+\frac{4}{5}$10(rectangles)+12 (rectangles)= 22 small rectangles. But each small rectangle = $\frac{1}{15}^{th}$ of $ABCD$. Therefore, $\frac{2}{3}+\frac{4}{5}=\frac{22}{15}$,which really means $\frac{2}{3}$+ of $ABCD$+$\frac{4}{5}$of $ABCD = \frac{22}{15}$. Similarly, $\frac{2}{3}+\frac{4}{5}$ is also $\frac{22}{15}$.

Now let us see what is $\frac{4}{5} - \frac{2}{3}$. From the Fig. 4, $\frac{4}{5}-\frac{2}{3}$=12 small rectangles -10

small rectangles = 2 small rectangles $=\frac{2}{15}$. There is no need to talk about the common denominator. We see the result using geometry. Later, we will consider many more examples to elucidate this point. Now let us consider multiplication.

Multiplication

Let us go back to Fig. 1. Clearly we see $2\times\frac{1}{2}=$ Two of $\frac{1}{2}=1$.

We also notice $2\times\frac{1}{4}$ Two of $\frac{1}{4}=\frac{1}{2}$. Similarly, (since $AGHD=\frac{4}{5}$th of $ABCD$ from Fig. 4) $\frac{2}{3}\times\frac{4}{5}=\frac{2}{3}$ of $\frac{4}{5}\times\frac{2}{3}$rd of the rectangle $AGHD$. Clearly, from Fig. 4, $AGIJ=\frac{1}{3}$rd of $AGHD$ and $AGKF=\frac{2}{3}$rd of $AGHD$ Therefore, $\frac{2}{3}\times\frac{4}{5}=AGKF$. Now counting the small rectangles in $AGKF$, we see there are 8 such rectangles, each rectangle being $\frac{1}{15}$ of $ABCD$. That means, $\frac{2}{3}\times\frac{4}{5}=\frac{2}{3}$ of $\frac{4}{5}=\frac{8}{15}$. If we consider $\frac{4}{5}\times\frac{2}{3}=\frac{4}{5}$th of $\frac{2}{3}$ then first we see what is $\frac{2}{3}$rd of $ABCD$. $\frac{2}{3}$ of $ABCD = ABEF$ $\frac{4}{5}\times\frac{2}{3}=\frac{4}{5}$th of $\frac{2}{3}=AKGF=\frac{8}{15}$. Then $\frac{2}{3}\times\frac{4}{5}=\frac{4}{5}\times\frac{2}{3}=\frac{8}{15}$. This concept could easily be extended to other fractions.

Division

Division is apparently the simplest of all. $\frac{\frac{2}{3}}{\frac{4}{5}}=\frac{ABEF}{AGHD}=\frac{\text{10 small rectangles}}{\text{12 small rectangles}}$ $=\frac{10}{12}$. Now the question is: What is $\frac{10}{12}$? Let $ABCD$ be a rectangle. Let us draw horizontal lines to divide it into 12 equal parts. $ABEF=FEHG=\frac{1}{12}$th of $ABCD$. Then, $ABHG=\frac{1}{6}$th of $ABCD$.From the figure it is clear that 10 slices of $\frac{1}{12}$th = 5 slices of $\frac{1}{6}$th. Therefore, $\frac{10}{12}=\frac{5}{6}$. This explains why, if we write, $\frac{10}{12}$ $=\frac{5\times2}{6\times2}$, the common factor 2, appearing both in the numerator and in the denominator could be cancelled, giving, $\frac{\frac{2}{3}}{\frac{4}{5}}=\frac{10}{12}=\frac{5}{6}$.

This also means that in a fraction both numerator and denominator may be multiplied by the same number without altering the value of the fraction.

Using this logic, we get immediately $\frac{1}{\frac{4}{5}}=\frac{5\times\frac{1}{5}}{4\times\frac{1}{5}}=\frac{5}{4}$ which is the reciprocal of $\frac{4}{5}$.
In the next section we will explain this geometrically.

Finally, let us see what is $\frac{2}{3}$ of $\frac{5}{4}$. In $ABCD$ there are 4 horizontal slices of $\frac{1}{4}$. In $ABXY$ there are 5 slices of $\frac{1}{4}$. Thus, $ABXY=\frac{5}{4}$. Slicing $ABXY$ vertically into 3 equal slices, we get $AEFY=\frac{2}{3}$ of $ABXY$.

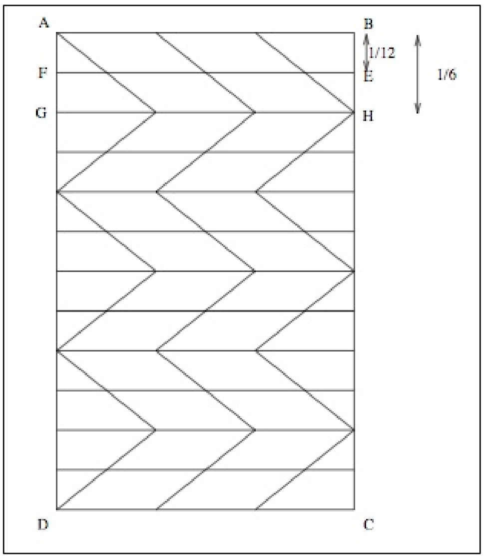

Figure 5. **Fraction 5.**

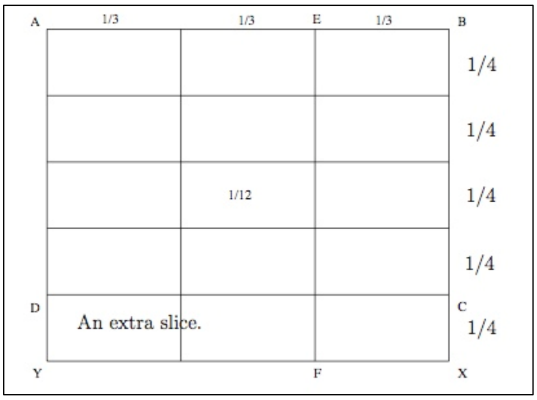

Figure 6. **Fraction 6.**

Now count all the small rectangles in $AEFY$. There are 10 such rectangles. Now remember that (not $ABXY$, but) $ABCD$ makes 1. In $ABCD$ there are 4 rows and 3 columns of small rectangles. Thus $4\times 3=12$ small rectangles are there. Therefore, each small rectangle = 1/12th of $ABCD$. Then

$\frac{2}{3}\times\frac{4}{5}=\frac{2}{3}$ of $\frac{5}{4}$=10 rectangles, each is $\frac{1}{12}$th of $ABCD$ $=10\times\frac{1}{12}=\frac{10}{12}=\frac{5}{6}$ (1)

If we now reconsider division of $\frac{2}{3}$ by $\frac{4}{5}$, (from the previous example) then

$$\frac{\frac{2}{3}}{\frac{4}{5}}=\frac{10}{12}=\frac{5}{6} \quad (2)$$

Therefore, from (1) and (2) we get $\frac{\frac{2}{3}}{\frac{4}{5}}=\frac{5}{6}=\frac{2}{3}\times\frac{5}{4}$ That means to divide a fraction by another fraction, we multiply the numerator by the reciprocal of the denominator. Now let us understand the concept of reciprocal geometrically.

Reciprocal

Let us understand what is $\frac{1}{\frac{4}{5}}$. We consider the rectangle *ABCD* as one rectangular object, like one pizza, as before.

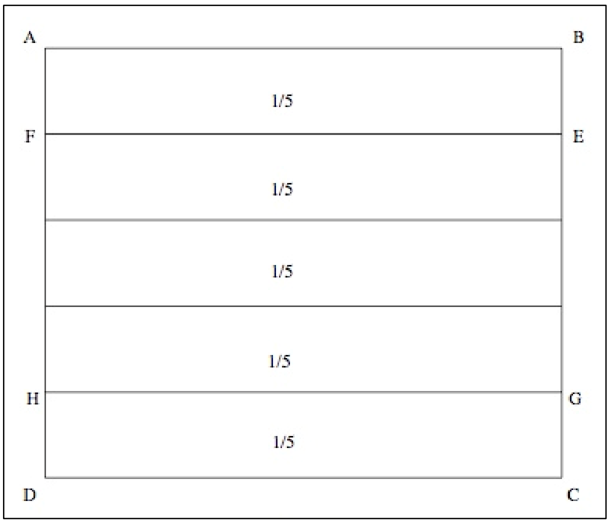

Figure 7. **Fraction 7.**

In the rectangle *ABCD* there are 5 equal horizontal slices, thus each slice $=\frac{1}{5}$. In *ABGH*, there are 4 slices. Therefore, from our previous ffs, $\frac{1}{\frac{4}{5}}=\frac{ABCD}{ABGH}=\frac{5 \text{ slices}}{4 \text{ slices}}=\frac{5}{4}$. $\frac{5}{4}$ is the reciprocal of $\frac{4}{5}$. Thus when we divided $\frac{2}{3}\div\frac{4}{5}$ and we expressed this as $\frac{2}{3}\times\frac{5}{4}$, it means we took the product of $\frac{2}{3}$ and the reciprocal of $\frac{4}{5}$.

More Illustrations

Add $\frac{3}{4}+\frac{5}{6}$; Subtract $\frac{5}{6}-\frac{3}{4}$; Multiply $\frac{3}{4}\times\frac{5}{6}$; Divide $\frac{3}{4}\div\frac{5}{6}$.

Solutions: *ABCD* is a rectangle divided in 6 equal parts vertically and 4 equal parts horizontally. Thus *ABCD* consists of 24 small rectangles.

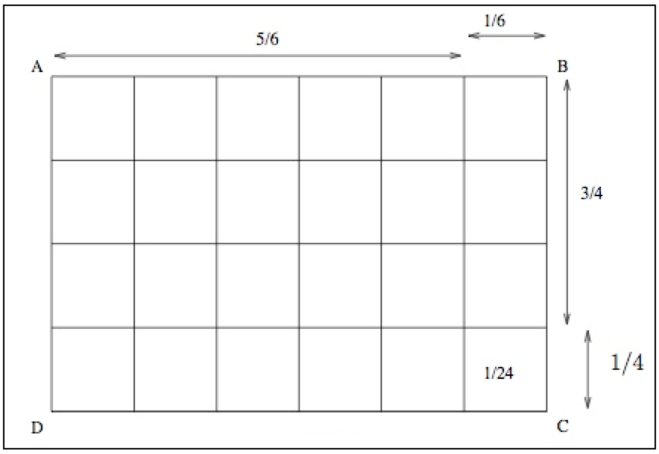

Figure 8. **Fraction 8.**

In $\frac{5}{6}$, there are $5\times4=20$ small rectangles, and in $\frac{3}{4}$, there are $3\times6=18$ small rectangles. Each rectangle is $\frac{1}{24}$. Thus following the technique of addition described before, $\frac{3}{4}+\frac{5}{6}=\frac{18+20}{24}=\frac{38}{24}=\frac{2\times19}{2\times12}=\frac{19}{12}=1\frac{17}{12}$(following section 2) Then following the technique of subtraction, $\frac{5}{6}-\frac{3}{4}=\frac{20-18}{24}==\frac{2}{24}=\frac{1\times2}{12\times2}=\frac{1}{12}$.

Following the principle of multiplication, $\frac{3}{4}\times\frac{5}{6}=\frac{3}{4}$ of $\frac{5}{6}=15$ small rectangles, each $=\frac{1}{24}=\frac{15}{24}=\frac{3\times5}{3\times8}=\frac{5}{8}$, For division $\frac{\frac{3}{4}}{\frac{5}{6}}=\frac{18}{20}=\frac{9}{10}$. From geometry of fractions it has become evident that $\frac{4}{7}+\frac{3}{5}=\frac{4\times5+3\times7}{7\times5}=\frac{20+21}{35}=\frac{41}{35}$ and $\frac{4}{7}+\frac{3}{14}$ $=\frac{4\times14+3\times7}{7\times14}=\frac{56+21}{7\times14}=\frac{77}{71\times4}=\frac{7\times11}{7\times14}=\frac{11}{14}$ where 14 is the common denominator. Thus the common denominator appears at the end and children do not have to hunt for it like hidden easter eggs at the outset. At this point, they may not have to draw rectangles to comprehend these arithmetics. But they will have a full grip on these concepts and should be able to explain to themselves and to others logical analysis of arithmetics of fractions using rectangles.

The geometry, discussed here, may be easily generalized to consider arithmetics of more than two fractions, although they may appear to be somewhat more complex, but hopefully not confusing. Children may look into these assignments as brain teasers or intellectual challenges.

For algebraic applications, we can translate geometry of fractions as follows:

1. $\frac{a}{b}+\frac{c}{d}=\frac{ab+bc}{bd}$

2. $\frac{a}{b}-\frac{c}{d}=\frac{ab-bc}{bd}$

3. $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$, and,

4. $\frac{(a/b)}{(c/d)}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$.

Also, if we have to add $\frac{a}{b}+\frac{c}{bd}=\frac{abd+bc}{b\cdot bd}=\frac{b(ad+c)}{b\cdot bd}=\frac{ad+c}{bd}$ where, the common denominator (bd) appears automatically at the end. According to our experience as teachers of mathematics, not only school children, sometimes even college students have to struggle hard to figure out common denominators when they deal with addition/subtraction of algebraic fractions. We sincerely hope that the methods explained in this article will bring some relief to them.

Discussions

On the Internet, as well as in several articles, we have noticed almost frantic efforts when teachers attempt to teach fractions to children. Some math teachers even said that fractions could be reduced to decimal numbers that children could use easily. This will be dangerous in our opinion. Because when they face algebraic fractions, they will encounter many more complications if they did not comprehend the fundamental rules of the arithmetic of fractions. Our objective is very simple and straightforward. Our mode of teaching consists of the following steps:

1. Understand visually, geometrically what fractions are.
2. Understand visually, geometrically what the rules of fractional arithmetic are. Master these rules.
3. Start using traditional methods. They are now more meaningful and easy to remember.

We do not think that traditional methods of teaching fractions, which have been in vogue for hundreds of years, should be completely replaced by other methods. These methods have a track record of success all over the world. We believe that new methods of teaching fractions will make the traditional method all the more successful. No one should say that our traditional techniques of teaching are not effective because they need memorization. This is not acceptable. We need some amount of memorization regardless what method we choose to learn fractions. What we need is a synthesis of older ideas with newer ideas and the teacher should do that.

For instance, in the references (Ho, 2008) we noticed several innovative ideas to make learning of fractions fun. About thirty years ago, we used a computer program of our own using various sounds and colors, coloring cookies, candies, and pizzas on a computer to teach our children fractions. The colors slowly lose their brightness and sounds lose their volumes as a big pizza for four is being divided into many parts. The smaller the part, less bright is the color and less volume of the sound. We also introduced similar ideas like fractions could be delicious by cutting fruit cakes into four pieces. Our son, Charlie, often will break his own piece of cake into two equal parts and explained to us that ½ of his ¼ piece of cake is really $1/8^{th}$ of the entire cake and it was fun for all of us to see this. Thus fractions could be a game too and interestingly, children could invent all such games and bring fun in fractions.

We believe that teachers should inspire children to play games that they can invent themselves. To explain to our children why in a fraction, denominator cannot be zero, we took a pizza and cut it into two pieces. They understood each piece is ½. I said it means that the number in the denominator shows how many equal pieces of pizza were there. If I want a piece which 1/0: Charlie was then 10, and he immediately responded “that will be meaningless daddy because you cannot cut this pizza into zero number of pieces and take one of those pieces.” Then he turned around and asked me: “Can you give me 3/2 of the cake?” I said: “Tell me how I could do it.” Then he answered his own question. He said, “Think of three pizzas. Cut that into two equal pieces. Therefore the answer should be 1 ½.” We are sure that teachers could invent, and encourage students to invent many such games, to make learning of fractions fun.

Finally, we must keep in mind that all these streams of thought will finally merge with the traditional system of learning fractions because that will be needed in algebra. We cannot abandon the traditional concepts of `common denominators’, `cross multiplication’, inequalities of fractions, etc. Some of our critics did mention it to us, that at this point, children should be taught how to use softwares like Maple, Mathematica, etc. We believe that in these days of advanced technology, students should be exposed to these softwares and they should learn them to apply at an advanced level in their respective disciplines. However, all the fundamentals of mathematics must be thoroughly learned first.

Conclusion

The concepts on geometry of fractions discussed in this article should be known to all teachers of mathematics. As such we claim no credit introducing them in this article. What we have tried to demonstrate are the applications of these concepts for visual understanding of arithmetic of fraction. We think that if teachers could use some colorful figures, students will be more attracted to learn these concepts and learning fractions will be truly a fun.

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